

Supplementary Material for "On the phase transition of light in cavity QED lattices"

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INTEGRATING OUT THE PHOTONS: EFFECTIVE SPIN-SPIN INTERACTIONS

An interesting perspective on the physics of a model of localized spins and itinerant photons can be obtained by integrating out the photonic degrees of freedom. This can be done exactly as the photons always enter quadratically. Let us first rewrite the quadratic part of the action associated to the Rabi model (we consider this case for simplicity, extension to the generalized Rabi is straightforward). After diagonalizing the hopping in Fourier space we get

$$S_0 = \int d\tau \sum_{\mathbf{k}} a_{\mathbf{k}}(\tau) (\partial_{\tau} - \omega_{\mathbf{k}}) \bar{a}_{\mathbf{k}}(\tau) = \quad (1)$$

$$\int d\tau \sum_{\mathbf{k}} a_{\mathbf{k}}(\tau) \mathcal{G}_{\mathbf{k}}^{-1}(\tau - \tau') \bar{a}_{\mathbf{k}}(\tau') \quad (2)$$

where the photonic dispersion reads $\omega_{\mathbf{k}} = \Omega - 2J \sum_{\alpha} \cos k_{\alpha}$ and we have introduced the bare photon Green's function

$$\mathcal{G}_{\mathbf{k}}(\tau) = -\langle T_{\tau} a_{\mathbf{k}}(\tau) a_{\mathbf{k}}^{\dagger}(0) \rangle \quad (3)$$

The coupling between the photon and the spin is linear in the field $x_i = a_i + \bar{a}_i$

$$S_g = g \int_0^{\beta} d\tau \sum_i x_i(\tau) \sigma_i^x(\tau) \quad (4)$$

which after Fourier transforming reads

$$\begin{aligned} S_g &= g \int_0^{\beta} d\tau \sum_{\mathbf{k}} x_{\mathbf{k}}(\tau) \sigma_{-\mathbf{k}}^x(\tau) = \\ &= g \int_0^{\beta} d\tau \sum_{\mathbf{k}} a_{\mathbf{k}}(\tau) \sigma_{-\mathbf{k}}^x(\tau) + a_{\mathbf{k}}^{\dagger}(\tau) \sigma_{\mathbf{k}}^x(\tau). \end{aligned} \quad (5)$$

The integration over the photon field can be done exactly by a simple shift of variable and the result reads

$$Z = \int D a D \bar{a} D \sigma e^{-(S_0 + S_g + S_{loc})} = \int D \sigma e^{-S_{eff}} \quad (6)$$

with a spin-only effective action of the form

$$S_{eff} = S_{loc} + g^2 \int d\tau d\tau' \sum_{\mathbf{k}} \sigma_{-\mathbf{k}}^x(\tau) \mathcal{G}_{\mathbf{k}}(\tau - \tau') \sigma_{\mathbf{k}}^x(\tau') \quad (7)$$

where $S_{loc} = \omega_q \sum_i \sigma_i^z$ is proportional to the total inversion of the TLSs, which acts as a transverse field. We can rewrite this effective action as following

$$S_{eff} = S_{loc} + \int d\tau d\tau' \sum_{\mathbf{k}} \sigma_{-\mathbf{k}}^x(\tau) J_{\mathbf{k}}(\tau - \tau') \sigma_{\mathbf{k}}^x(\tau') \quad (8)$$

where the effective spin-spin interaction is given in terms of the Green's function of the photon field

$$J_{\mathbf{k}}(\tau) = \frac{g^2}{2} (\mathcal{G}_{\mathbf{k}}(\tau) + \mathcal{G}_{\mathbf{k}}(-\tau)) \quad (9)$$

We notice this coincides with the Matsubara Green's function of the photon displacement

$$J_{\mathbf{k}}(\tau) = -\frac{g^2}{2} \langle T_\tau x_{\mathbf{k}}(\tau) x_{-\mathbf{k}}(0) \rangle \quad (10)$$

a result which is also expected by looking at the order by order perturbation theory.

THE RABI LATTICE MODEL WITH THE A^2 TERM

In this section we discuss the modification of the phase boundary of the Rabi Lattice model when the A^2 term of the form

$$H_{A^2} = D \sum_{\mathbf{R}} (a_{\mathbf{R}} + a_{\mathbf{R}}^\dagger)^2 \quad (11)$$

is added. For the sake of simplicity we consider the case $g' = g$, hence the full Hamiltonian reads

$$\mathcal{H} = -J \sum_{\langle \mathbf{R}\mathbf{R}' \rangle} a_{\mathbf{R}}^\dagger a_{\mathbf{R}'} + \sum_{\mathbf{R}} \mathcal{H}_R[a_{\mathbf{R}}, \sigma_{\mathbf{R}}^+] + H_{A^2} \quad (12)$$

where $\mathcal{H}_R$ is the hamiltonian of the single site Rabi model. We start noticing that, at a general level, this extra term does not break explicitly the Z_2 symmetry of the problem, hence it doesn't *a priori* rule out the existence of the phase transition. At the same time we expect this term to modify the phase boundary in such a way to favour the disordered (symmetric) phase.

We notice at this point that, contrarily to the case of single mode Dicke Model, our lattice Hamiltonian cannot be solved exactly in general. In order to proceed it is useful to perform a canonical transformation on the photon fields to eliminate H_{A^2} . This can be done by expressing the photon operators $a_{\mathbf{R}}, a_{\mathbf{R}}^\dagger$ on each lattice site in terms of a new set of degrees of freedom $b_{\mathbf{R}}, b_{\mathbf{R}}^\dagger$

$$a_{\mathbf{R}} = \cosh\theta b_{\mathbf{R}} + \sinh\theta b_{\mathbf{R}}^\dagger \quad (13)$$

$$a_{\mathbf{R}}^\dagger = \sinh\theta b_{\mathbf{R}} + \cosh\theta b_{\mathbf{R}}^\dagger \quad (14)$$

By choosing

$$\tanh 2\theta = -2D / (\omega_r + 2D)$$

we can eliminate the A^2 term and obtain a local Hamiltonian which is again of the Rabi form, yet with renormalized photon parameters

$$\tilde{H}_{rabi} = \sum_{\mathbf{R}} \tilde{\omega}_r b_{\mathbf{R}}^\dagger b_{\mathbf{R}} + \frac{\omega_q}{2} \sigma_{\mathbf{R}}^z + \tilde{g} \sigma_{\mathbf{R}}^x (b_{\mathbf{R}}^\dagger + b_{\mathbf{R}}) \quad (15)$$

where

$$\tilde{\omega}_r = \sqrt{\omega_r^2 + 4D\omega_r} \quad \tilde{g} = g\sqrt{\omega_r/\tilde{\omega}_r}$$

We further notice that the hopping term between different resonators gets also renormalized and in addition a pair-hopping term arises so that the hopping Hamiltonian reads in terms of the new fields

$$\tilde{H}_{hopping} = -\tilde{J} \sum_{\langle \mathbf{R}\mathbf{R}' \rangle} (b_{\mathbf{R}}^\dagger b_{\mathbf{R}'} + hc) + \quad (16)$$

$$-\tilde{J}_{pair} \sum_{\langle \mathbf{R}\mathbf{R}' \rangle} (b_{\mathbf{R}} b_{\mathbf{R}'} + b_{\mathbf{R}}^\dagger b_{\mathbf{R}'}^\dagger) \quad (17)$$

where the renormalized hopping read respectively

$$\tilde{J} = J \frac{\omega_r + 2D}{\tilde{\omega}_r} \quad \tilde{J}_{pair} = -J \frac{2D}{\tilde{\omega}_r} \quad (18)$$

We start noticing that, quite generically for $D > 0$, the pair hopping term satisfies the condition

$$\frac{|\tilde{J}_{pair}|}{\tilde{\omega}_r} = \left(\frac{2D/\omega_r}{1 + 4D/\omega_r} \right) \frac{J}{\omega_r} < \frac{J}{\omega_r} \quad (19)$$

from which we conclude that if we assumed in the original problem $J \lesssim \omega_r$ (an assumption which is behind the Hamiltonian 12) then, within the same accuracy, we can drop the pair hopping terms in the transformed hamiltonian. From the above discussion we conclude that the effect of the A^2 term can be fully encoded in a new lattice Rabi model

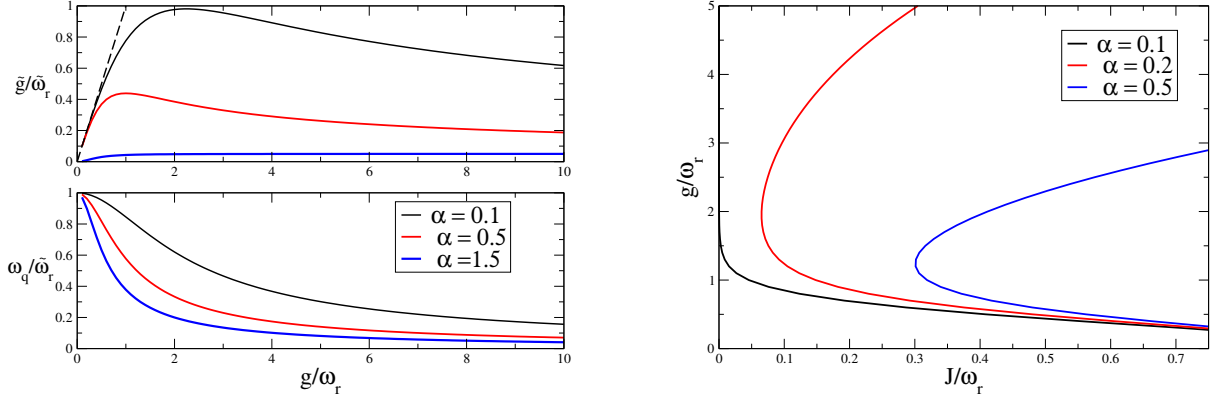


FIG. 1: Left Panel: Renormalized light-matter coupling $\tilde{g}/\tilde{\omega}_r$ and TLS frequency $\omega_q/\tilde{\omega}_r$ as a function of the bare coupling g/ω_r for different values of the parameter α . We notice that for small g/ω_r the effects of renormalization are negligible for any α and become more relevant for larger values of g/ω_r . Right Panel: Mean Field Phase diagram for different values of the parameter α .

with renormalized parameters. Whenever the coupling D can be made independent of the light-matter coupling g , a situation that can arise in certain circuit QED architectures, the effect of the A^2 term is just a trivial uniform shift of the phase boundary in the g vs J plane. The opposite situation is more tricky. We will assume in the following a coupling D that depends on the light-matter interaction g as $D = \alpha \frac{g^2}{\omega_q}$ and discuss the physics of the model for different values of α . In order to gain insight, it is useful to look at how the coupling D renormalizes the photon parameters. In Figure 1 we plot for different values of α the renormalized effective light-matter coupling $\tilde{g}/\tilde{\omega}_r$ and the renormalized effective TLS frequency $\omega_q/\tilde{\omega}_r$ as a function of g/ω_r . We see that in general the role of a finite α is (i) to scale down the effective light-matter interaction particularly in the ultrastrong coupling regime $g/\omega_r \gg 1$, and (ii) to renormalize down the TLS frequency $\omega_q/\tilde{\omega}_r \ll 1$, toward a regime of large detuning. We notice that the two effects compete with each other, as the former reduces the tendency to order while the latter enhances it. This competition, in the case of *single-mode* Dicke model, has a rather drastic consequence for sufficiently large α – the disappearance of the superradiant critical point. However, while the phase transition in the Dicke model is driven only by the local physics, namely g vs. ω_q , in the Rabi model increasing the hopping strength J is another route to access the super-radiant phase. For this we find that the hopping strength is only weakly affected by the renormalization due to D . It is easy to see using Eq. (18) and the definition of D that $\tilde{J}/\tilde{\omega}_r \sim J/\omega_r$, even at ultra-strong coupling. In conclusion, we expect the critical point to be accessible even in presence of the A^2 term. While a thorough discussion of this issue will be presented elsewhere, here we give support to this claim by solving the Rabi Hubbard model with the A^2 term within the Gutzwiller mean field approximation. As we see from the right panel of Figure 1 the shape of the phase boundary is only quantitatively affected by finite α , particularly in the ultra-strong coupling regime, but the overall picture is not changed, namely a hopping driven quantum phase transition between a disordered normal phase and a delocalized super-radiant phase where the TLSs order ferromagnetically.