

Stabilizing entanglement via symmetry-selective bath engineering in superconducting qubits *Supplementary Information*

DETAILED EXPERIMENTAL SETUP

There are three principal drives used in this experiment (Figure 1), generated by APSIN and Agilent microwave generators. The readout drive (purple block) is mixed at DC with readout pulses generated by a marker output of a Tektronix AWG520 two-channel arbitrary waveform generator (AWG); amplified to achieve sufficient power for high-power single-shot readout [1]; gated by a fast switch to avoid excessive qubit dephasing from readout leakage; and sent through a LabBrick digital attenuator for fine-tuning of readout amplitude. The readout is combined with qubit and cooling drive pulses that are input at cavity *B*. Additionally, half of the generator output signal serves as the local oscillator (LO) for an IQ demodulator for homodyne detection of the readout signal.

The qubit drives are generated by a microwave source at 6.075 GHz. This drive serves as the LO for two single-sideband modulators (SSBs) optimized for upper sideband signal output. The two channel outputs of the AWG generate qubit rotation pulses at 125 MHz, with *X*- and *Y*- rotations defined by the relative starting phases of the pulses; these are mixed up to 6.2 GHz, the operating frequency of the qubits. Since the qubits are resonant, the difference in path length between the generator and the two cavity inputs is important; we describe the calibration of this angle in the next section. Qubit pulse amplitudes are calibrated via the AWG.

The cooling drive is split in two: the tone that is sent to cavity *A* passes through a digital attenuator for amplitude balance, and the drive *en route* to cavity *B* passes through a digital phase shifter to enable facile manipulation of the symmetry of the drives.

The timing of the experiment is controlled by an SRS DG535 digital delay generator (not shown). the DG535 serves three purposes: it gates the cooling drive generator, setting the start and length of the cooling operation; it triggers the AWG to perform qubit rotations and readout; and it triggers the Alazar digital acquisition card to accept input data.

The qubit, cavity, and cooling drives are thermalized to base by 20-, 10-, and 20- dB attenuators thermalized to the 4 K, 1 K, and 30 mK stages of a closed-cycle He-3 dilution refrigerator. At base, they are further filtered for stray infrared (IR) by homemade lossy Eccosorb filters, and input into a secondary copper containment canister containing the coupled cavity system. This can is indium-sealed and coated with further IR-absorbing material. The coupled-cavity output passes through three cryogenic isolators and a commercial K&L low-pass filter at base. It is then amplified by a Low Noise Factory HEMT at 4 K and by two room-temperature amplifiers before being demodulated and digitized.

FULL SYSTEM CALIBRATION

The full set of relevant qubit and cavity parameters include the coupled cavity mode frequencies ($\omega_{\pm}^c$); the cavity-cavity coupling (J); the qubit anharmonicities (α_i); the qubit-cavity couplings (g_i); the symmetric and antisymmetric cavity linewidths ($\kappa_{\pm}$); and the qubit lifetimes ($T_{1,i}$) and dephasing times ($T_{\phi,i}$). These parameters are tabulated in Table I.

Coupling of the cavities

The cavities are designed with dimensions of 1.20" x 0.20" x 1.00", with the long axis terminated cylindrically at both ends with a diameter of 0.20 inches (see Figure 1(a) of the main paper). HFSS modal analysis predicts these cavities to have a TE₁₀₁ mode at 7.2 GHz in the absence of a qubit chip. The measured resonator frequency in the presence of a qubit chip is 7.114 GHz at base temperature; the silicon chip lowers the cavity frequency due to the silicon dielectric. The two cavities are machined in HOFC copper and are coupled by an aperture on the transverse axis of width 0.04" and height 0.52". HFSS modeling again predicts that this aperture will couple the two cavity modes with a splitting $2J = 290$ MHz, where the magnitude of J is highly sensitive to the machining tolerances of the cavity. This compares well to $2J = 280$ MHz that we measure experimentally.

Since we cannot separate the cavities to directly measure their bare frequencies, we manually fine-tune the cavity frequencies by referring to the avoided crossing between them, which results in two coupled cavity modes at $\omega_{\pm}^c =$

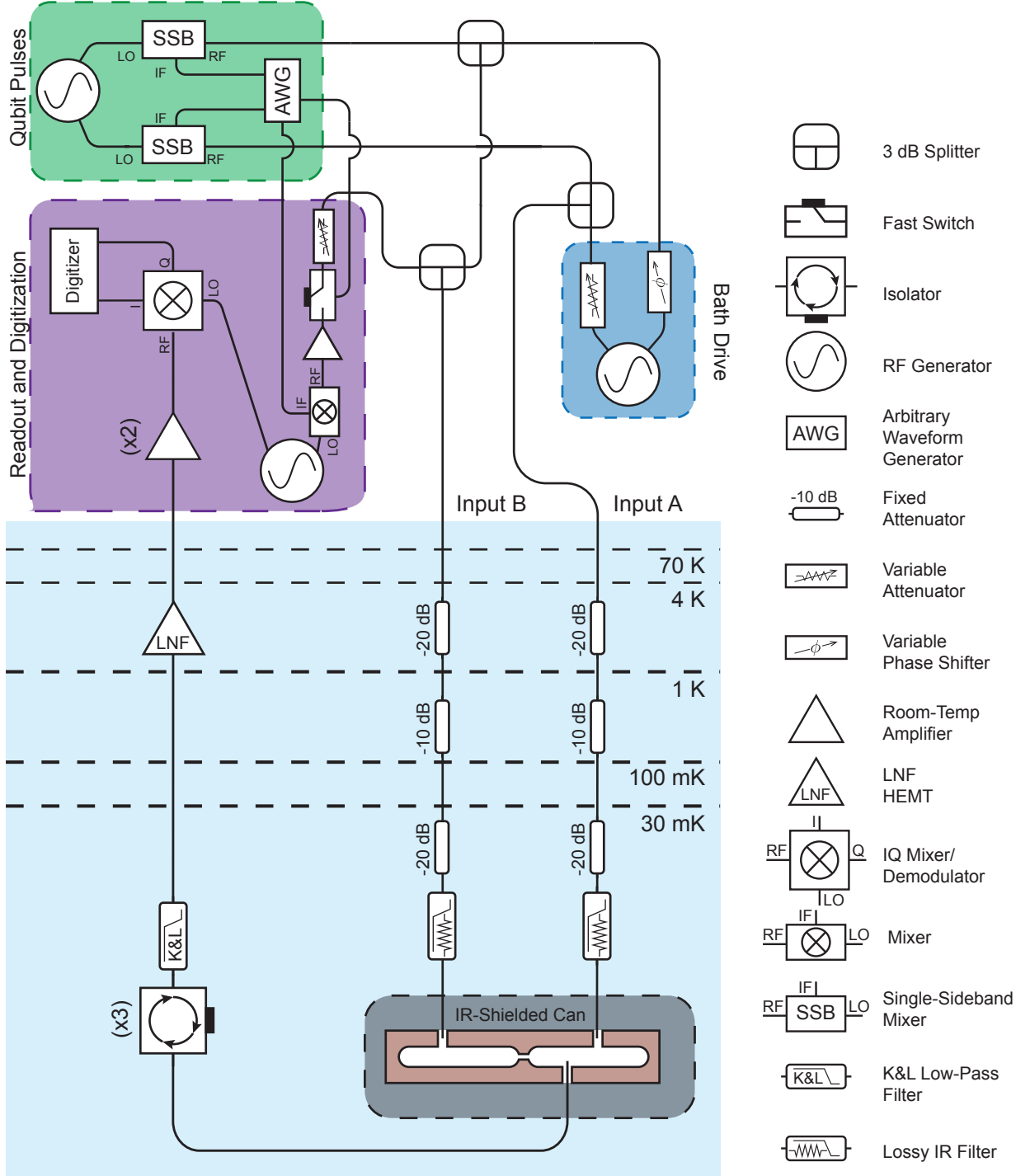


FIG. 1: Experimental setup. Not shown is an SRS DG535 Digital Delay Generator, which is responsible for triggering the AWG (which generates qubit rotations and gated readout pulses), the cooling drive generator, and the Alazar digital acquisition card.

$\frac{1}{2}(\omega_A^c + \omega_B^c) \mp \frac{1}{2}\sqrt{(\omega_A^c - \omega_B^c)^2 + 4J^2}$. The splitting between the two modes is minimized when the cavities are exactly on resonance ($\omega_A^c = \omega_B^c$). By slightly adjusting the positions of the qubit chips and employing standard S_{21} transmission measurements to find the coupled cavity modes, we minimize this splitting and thus tune the cavities to resonance. These fine adjustments are on the order of millimeters, and correct for minor inhomogeneities in machining as well as for chip size mismatch. The coupling of the cavities shifts the spatial mode structure of the symmetric and

$\omega_+/2\pi$	6.9524 GHz
$\omega_-/2\pi$	7.2310 GHz
$\kappa_+/2\pi$	650 kHz
$\kappa_-/2\pi$	820 kHz
$\alpha_A/2\pi$	-210 MHz
$\alpha_B/2\pi$	-210 MHz
$J/2\pi$	139.3 MHz
$g_A/2\pi$	89 MHz
$g_B/2\pi$	98 MHz
$T_{1,A}$	4 μs
$T_{1,B}$	7.2 μs
$T_{2,A}$	2.6 μs
$T_{2,B}$	3 μs

TABLE I: Calibrated experimental parameters.

antisymmetric modes, leading to unequal photon decay rates $\kappa_{\pm}$ via the strongly-coupled output port.

Note that, because the cavities are nominally identical, there are additional avoided crossings between the higher-order cavity modes, which similarly hybridize into symmetric and antisymmetric modes. Both the resonant frequencies and the coupling between these modes are highly sensitive to machining errors, and in general the HFSS modeling predicts these frequencies less accurately than the fundamental mode. However, because these modes are above 9 GHz, they couple minimally to the qubits, and because they are far-detuned from any qubit or bath drives, we do not expect them to participate in the cooling protocol.

Qubit Parameters

The qubit-cavity couplings g_i are calibrated by measuring the cross-Kerr nonlinearity of the antisymmetric mode (χ_-) as a function of qubit-cavity detuning. We do so by performing two-tone qubit spectroscopy (see Figure 1c of the main paper) at a slightly higher readout power, such that the photon number occupation of the measurement mode (ω_-^c , in this instance) is equal to $\bar{n} \approx 0.5$. The experiments are performed in the photon number-resolved limit ($\chi > \kappa, 1/T_2$), such that we can resolve two spectral lines at $\omega^q|_{n=0}$ and $\omega^q|_{n=1}$, which are split by $2\chi_-$. The splitting χ_- is related to g by

$$\chi_- = \frac{\alpha}{(\Delta_- + \alpha)} \frac{g^2}{\Delta_-}. \quad (1)$$

Here, α is the transmon nonlinearity, and $\Delta_- \equiv \omega^q - \omega_-^c$ is the detuning between the qubit and the antisymmetric cavity mode. We sweep the bias coils that independently govern the qubit frequencies and plot $|2\chi_-|$ as a function of $|\Delta_-|$ for both qubits independently. Fitting to Equation 1 provides the fit shown in Figure 2(c-d) and tabulated in Table I.

The bare (uncoupled) qubit coherence times T_1, T_2 are calibrated via standard T_1 and Ramsey measurement protocols. These measurements are performed for each qubit at the operating frequency of 6.2 GHz, while the qubit not under test is detuned by 50 MHz and is unaffected by qubit excitation pulses at 6.2 GHz. The Purcell limit is set by the photonic density of states at the individual qubit positions, summed over all the normal modes of the system. The qubits couple most strongly to the symmetric and antisymmetric combinations of the TE₁₀₁ modes, but the higher-order modes also participate. The total Purcell decay limit on T_1 considering only the lowest two modes is approximately 8-10 μs (although we note that the 3D multimode Purcell effect typically reduces Purcell decay when the qubits are red-detuned from the fundamental cavity mode due to interference between the modes).

The coherence times we achieve are shorter than state-of-the-art 3D transmons, although T_1 is near the single-mode Purcell limit. However, note that T_1 is built as a feature into our cooling protocol, as it depopulates unwanted qubit states, and T_{ϕ} , the pure dephasing rate that can be extracted from $\frac{1}{T_2} = \frac{1}{T_{\phi}} + \frac{1}{2T_1}$, is lengthened by a factor of 10 or more by the lifting of the degeneracy between the $|S\rangle$ and $|T_0\rangle$ states when the qubits are tuned to resonance, as

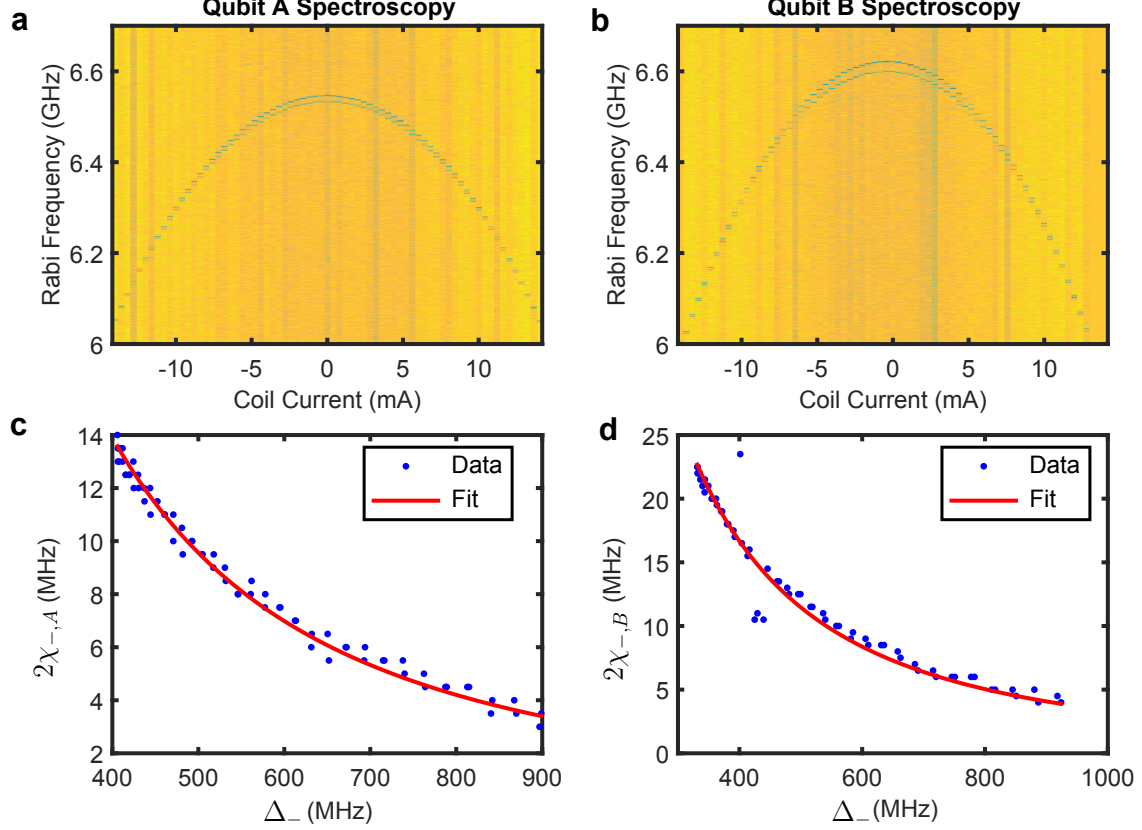


FIG. 2: Calibration of the qubit-cavity couplings g_i . We sweep the qubit frequency and measure the cross-Kerr nonlinearity of the antisymmetric cavity mode (χ_-) as a function of qubit-cavity detuning for qubit A (panels a, c) and B (panels b, d). This provides a fit of g_i as per Equation 1. The outlier points in Panel d result from an avoided crossing as qubit B tunes through qubit A .

seen in the coupled rate equation fits in Figure 4 of the main text. Lifting this degeneracy implies that the noise that couples these two states is no longer at DC, but is in fact the noise at $\omega = 2\delta$. This effect provides protection from low-frequency noise, and results in an increased effective phase coherence between $|S\rangle$ and $|T_0\rangle$.

Path Length Calibration

Because the qubits are tuned to degeneracy, the qubit pulses are performed in the same rotating frame. It is critical to properly calibrate the path length difference between pulses arriving at cavities A and B in order to ensure that pulses are performed along common X- and Y-axes. In principle, we can use our knowledge that the higher-energy state to which we cool using our bath engineering scheme is the $|S\rangle$ state, but we also must independently verify this hypothesis.

We follow the methods in the supplementary material of Ref. [2]: noting that the XY Hamiltonian drives oscillations between $|ge\rangle$ and $|ieg\rangle$, we perform a $R_X^{\pi/2}$ rotation on qubit A , along with a $R_\theta^{\pi/2}$ rotation about an arbitrary axis on qubit B , and then allow the coupled qubit state to freely evolve, performing tomographic reconstructions along the way in order to resolve the joint density matrix $\rho(\theta, t)$. When $\theta = \pi/2$ (that is, when we perform a qubit B rotation about the y-axis), the oscillation amplitudes between $\rho_{|ge\rangle\langle ge|}$ and $\rho_{|ieg\rangle\langle ieg|}$ (that is, the qubit populations in $|ge\rangle$ and $|ieg\rangle$, respectively) are maximized, and oscillations are minimized at $\theta = \{0, \pi\}$. We perform a first Rabi oscillation measurement at arbitrary θ ; we then fix the Rabi oscillation time to a maximum of $\rho_{|ge\rangle\langle ge|}$ and sweep θ . We add a constant phase offset to the qubit rotation pulses applied to qubit B such that the oscillations are suppressed at

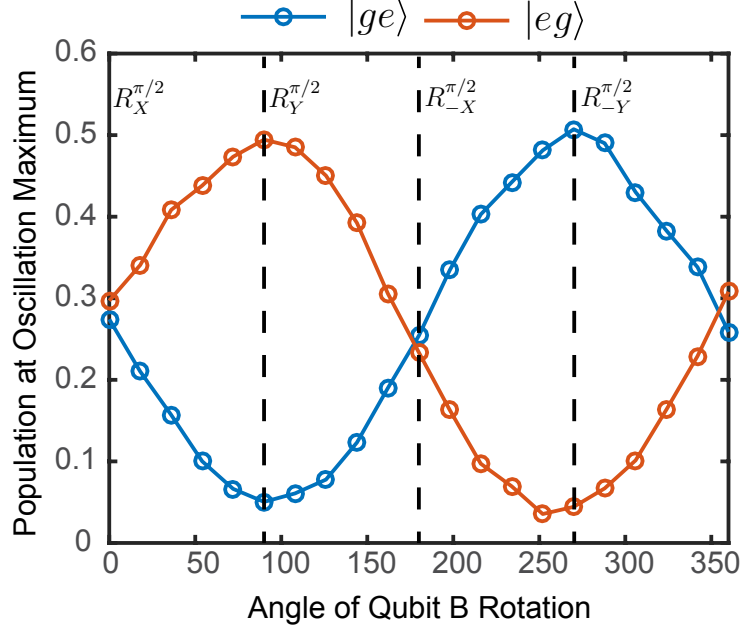


FIG. 3: Calibration of the relative path lengths of qubit pulses. We prepare the qubits using $R_X^{\pi/2}$ on qubit A and $R_\theta^{\pi/2}$ on qubit B , and fix the free evolution time to $\tau = 400$ ns, where we find a maximum in the oscillation amplitude between $\rho_{|ge\rangle\langle ge|}$ and $\rho_{|eg\rangle\langle eg|}$. We then vary θ and fix an offset such that the amplitude of oscillation is damped at $\theta = \{0, \pi\}$.

$\theta = \{0, \pi\}$ (Figure 3).

Balancing the Cooling Drive Amplitudes

Achieving symmetry-selective bath engineering in our experimental setup requires a careful calibration of the relative drive amplitudes inside the two cavities, such that the two-photon coherent driving rate is the same for both qubits. If the cavity input amplitudes are not properly balanced, then even when the phases are properly aligned (say, fully symmetrically), the excess drive in one of the cavities means that the decomposition of the total drive into symmetric and antisymmetric drives will have a nonzero amplitude in the undesired component (antisymmetric, in this case). This reduces the symmetry selectivity, so balancing the cavity amplitudes is critical. Quantitatively, this condition is met if $g_A^2 \bar{n}_A = g_B^2 \bar{n}_B$, or alternatively, if

$$\frac{g_A \sqrt{\kappa_A P_A^{in}}}{\omega^c - \omega^d} = \frac{g_B \sqrt{\kappa_B P_B^{in}}}{\omega^c - \omega^d}. \quad (2)$$

where we emphasize that P_i^{in} is a controllable parameter and is related to ϵ_i^d by $\kappa_i P_i^{in} = |\epsilon_i^d|^2$.

Several of these parameters are not experimentally measurable - in particular, we cannot independently extract the cavity decay rates κ_i because the resonances that we measure are hybrids of both cavity modes and are affected by both κ_A and κ_B . Therefore, we use pump-probe spectroscopy of the bath engineering drive to directly balance the drive amplitudes (Figure 4). We adjust the relative phase of the drives to $\phi = \pi$ such that the drive is fully antisymmetric; we then sweep the cooling drive across the expected frequencies for cooling to both $|S\rangle$ and $|T_0\rangle$. In Figure 4, we calibrate the drive amplitudes for cooling via ω_-^c , the antisymmetric cavity mode; in this case, we expect $|T_0\rangle$ to be a forbidden state when the amplitudes are properly balanced. We choose the digital attenuation setting that best suppresses $|T_0\rangle$ as our operating point (as indicated by the arrow).

TOMOGRAPHY

Tomography in this experiment is performed according to the method outlined in Chow et al. [3]. Because the cavity bandwidth is narrow, we use high-power readout [1] that is optimized to discriminate the joint ground

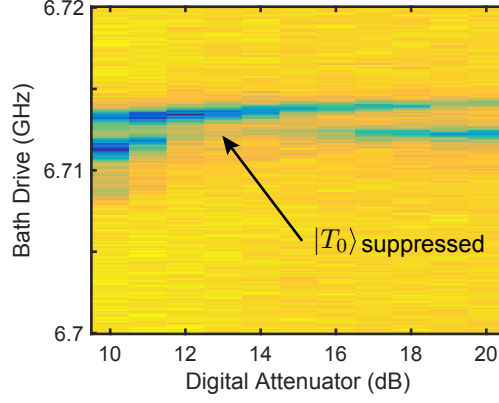


FIG. 4: Calibration of the relative cooling drive amplitudes. Here, we cool via ω_-^c and fix $\phi = \pi$, such that we expect $|T_0\rangle$ to be fully suppressed when the drive amplitudes are balanced. We perform spectroscopy of the cooling drive while varying the set point of the digital attenuator that controls the relative amplitude of the two cavity drives. Away from the ideal balance, both the upper and the lower cooling lines appear; however, when the drives are balanced (arrow), symmetry selectivity suppresses $|T_0\rangle$. The red-shifting of the cooling frequencies that occurs at lower attenuator settings is due to the AC Stark effect.

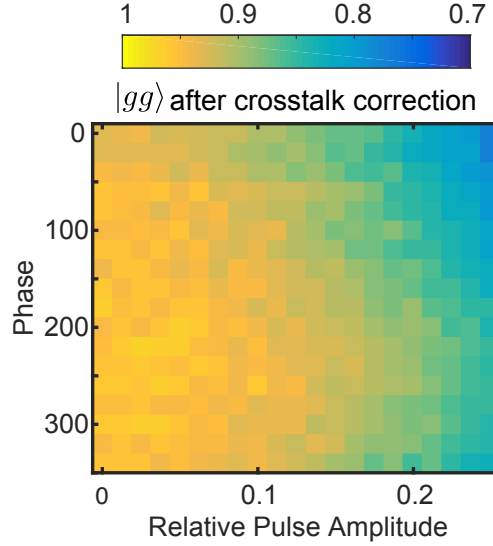


FIG. 5: Calibration of the qubit pulse cross-talk. We apply the equivalent of a π -pulse to Cavity A while Qubit A is detuned, and implicitly measure the effect on Qubit B by performing crosstalk correction pulses at swept relative amplitude (x-axis) and phase (y-axis). The point at which the ground-state population is maximized (yellow) indicates that the cross-talk rotation has been corrected. The color scale indicates the ground-state population.

state $|gg\rangle$ from the other three eigenstates of the system. This readout accomplishes a measurement of the form $\mathcal{M} = \beta_{II}\sigma_{II} + \beta_{IZ}\sigma_{IZ} + \beta_{ZI}\sigma_{ZI} + \beta_{ZZ}\sigma_{ZZ}$, where the β -coefficients are calibrated using the double-Rabi oscillation method in the previous reference. We perform a set of 30 (overspecified) positive and negative qubit rotations in order to fully determine the 15 degrees of freedom of the density matrix, and use maximum likelihood estimation to infer the trace-normalized, positive-semidefinite density matrix that is consistent with our tomographic data. In order to accomplish single-qubit pulses despite the resonant coupling of the qubits, we apply 32-ns qubit pulses such that the pulse envelope is broader than the qubit splitting, enabling single-qubit rotations. While the qubit coupling induces mixing between the $|ge\rangle$ and $|eg\rangle$ states, it does not affect the $|gg\rangle$ population from which we infer the density matrix; this limits the effect of the qubit coupling on tomographic reconstruction.

Finite-element simulations show that a pulse at the qubit frequency, when applied to a given cavity, remains localized in that cavity and does not propagate to the neighboring cavity. However, we would like to calibrate *in-situ* the qubit pulse cross-talk (i.e. the amount by which a rotation on Qubit A will also rotate Qubit B). To do so, we calibrate a

π -pulse on Qubit A, and then detune it by 50 MHz so that the pulse does not in fact affect the state of Qubit A. We then (Figure 5) apply compensating pulses to Qubit B at a swept amplitude and frequency. When the $|gg\rangle$ population is maximized (yellow), the crosstalk has been undone; this provides a direct calibration of the crosstalk. We see a slight phase-dependence in the data, indicating that the crosstalk occurs at a relative phase of 250° , and estimate that the amplitude of the crosstalk is less than 5%. The fact that we do both positive and negative tomographic rotations partially corrects for this crosstalk, as the two sets of rotations will induce opposite cross-talk.

THEORETICAL ANALYSIS

Model

Our experimental setup can be modeled by a pair of two identical two-level systems housed in two identical single-mode optical cavities which can exchange photons via coherent tunneling. Both cavities are driven out of equilibrium by two coherent microwave drives with same frequency ω^d detuned from the cavity frequency. In the frame rotating at ω^d , after a rotating-wave approximation (*i.e.* neglecting counter-rotating terms), the setup is well described by the following Hamiltonian [4]:

$$H = H_\sigma + H_{\sigma,a} + H_a, \quad (3)$$

with H_σ , $H_{\sigma,a}$, and H_a respectively being the qubit, the light-matter coupling, and the photon Hamiltonians given by (we set $\hbar = 1$)

$$H_\sigma = \sum_i (\omega^q - \omega^d) \frac{\sigma_i^z}{2}, \quad H_{\sigma,a} = \sum_i g_i \left[a_i^\dagger \sigma_i^- + a_i \sigma_i^+ \right], \quad H_a = \sum_i \left[(\omega^c - \omega^d) a_i^\dagger a_i + \epsilon^d (a_i^\dagger e^{i\phi_i} + \text{h.c.}) \right] - J(a_A^\dagger a_B + a_B^\dagger a_A). \quad (4)$$

The two-level systems are represented by Pauli matrices obeying $[\sigma_i^a, \sigma_j^b] = 2i\delta_{ij}\epsilon_{abc}\sigma_i^c$ where the indices $a, b, c = x, y, z$, the cavity indices $i, j = A, B$ and $\sigma_i^\pm \equiv (\sigma_i^x \pm i\sigma_i^y)/2$. ω^q is the energy splitting between ground and excited qubit states. $a_i^\dagger$ and a_i are respectively the cavity photon creation and annihilation operators, $[a_i, a_j^\dagger] = \delta_{ij}$. ω^c is the cavity frequency, the g_i 's are the light-matter coupling constants, and J sets the amplitude of the photon hopping between both cavities. ϵ^d and ω^d are respectively the amplitude and the frequency of both ac drives. ϕ_A and ϕ_B denote their corresponding phases. Given that physics only depends on the relative phase difference, we shall work with $\phi \equiv \phi_B - \phi_A$.

Let us neglect the light-matter coupling for a moment. On the qubit side, the eigen-states and eigen-energies of H_σ alone are given by the triplet and singlet states:

$$\begin{aligned} |T_+\rangle &\equiv |e\rangle_A \otimes |e\rangle_B \equiv |ee\rangle, & E_{|T_+\rangle} &= \Delta^{\text{qd}}, \\ |S\rangle &\equiv [|ge\rangle - |eg\rangle]/\sqrt{2}, & E_{|S\rangle} &= 0, \\ |T_0\rangle &\equiv [|ge\rangle + |eg\rangle]/\sqrt{2}, & E_{|T_0\rangle} &= 0, \\ |T_-\rangle &\equiv |gg\rangle, & E_{|T_-\rangle} &= -\Delta^{\text{qd}}, \end{aligned} \quad (5)$$

where $\Delta^{\text{qd}} \equiv \omega^q - \omega^d$ is the detuning between the qubits and the drive. On the photonic side, the two coupled cavities being identical, the eigen-modes of the undriven photonic backbone are naturally symmetric and antisymmetric excitations with the creation operators

$$A^\dagger \equiv \frac{a_A^\dagger + a_B^\dagger}{\sqrt{2}} \quad \text{and} \quad a^\dagger \equiv \frac{a_A^\dagger - a_B^\dagger}{\sqrt{2}},$$

respectively. In this basis, H_a reads

$$H_a = \Delta_+^{\text{cd}} A^\dagger A + \Delta_-^{\text{cd}} a^\dagger a + \sqrt{2}\epsilon^d [A^\dagger \cos(\phi/2) - ia^\dagger \sin(\phi/2)] + \text{h.c.}, \quad (6)$$

with $\Delta_\pm^{\text{cd}} \equiv \omega_\pm^c - \omega^d$ and $\omega_\pm^c \equiv \omega^c \mp J$.

We include the photon leakage outside the cavities with a rate κ_+ for the symmetric mode and κ_- for the antisymmetric mode. The density of states for the antisymmetric and symmetric modes are $\rho_\pm(\omega) = -\text{Im} G_\pm^{\text{R}}(\omega)/\pi$ where the retarded Green's function are given by, in the rotating frame, $G_\pm^{\text{R}}(\omega) = 1/[\omega - \Delta_\pm^{\text{cd}} + i\kappa_\pm/2]$. We also include qubit decay and pure dephasing rates, Γ_1 and Γ_φ respectively. These dissipative processes will play a paramount role in the cooling scheme and protocol.

Dynamics

Altogether, the driven-dissipative dynamics of the entire system (qubits and photons) is well described by the following Master equation on the density matrix ρ :

$$\dot{\rho} = -i[H, \rho] + \kappa_+ \mathcal{D}[A]\rho + \kappa_- \mathcal{D}[a]\rho + \Gamma_{1,A} \mathcal{D}[\sigma_A^-]\rho + \Gamma_{1,B} \mathcal{D}[\sigma_B^-]\rho + \frac{\Gamma_{\varphi,A}}{2} \mathcal{D}[\sigma_A^z]\rho + \frac{\Gamma_{\varphi,B}}{2} \mathcal{D}[\sigma_B^z]\rho, \quad (7)$$

where the Lindblad-type dissipators are defined as $\mathcal{D}[X]\rho \equiv (X\rho X^\dagger - X^\dagger X\rho + \text{h.c.})/2$. Note that Eq. (7) can be solved numerically and $\rho(t)$ accessed at all times. In particular, the knowledge of the steady-state density matrix $\rho^{\text{NESS}} = \lim_{t \rightarrow \infty} \rho(t)$ allows to compute the fidelities of interest. The results obtained from these numerical computations agree with the experimental results and are also a way to check the validity of the different layers of approximations that are performed in the analytic approach developed below.

Effective driven-dissipative XY model

We now present an analytic approach to compute the steady-state fidelities for the qubit subsystem. In a nutshell, it consists in eliminating the explicit time-dependence of the problem by means of a rotating-wave approximation, in treating the light-matter interaction by second-order perturbation theory, in linearizing the photonic degrees of freedom around mean-field solutions, and in integrating out the remaining photon-fluctuations by means of a Master-equation approach. The result is a set of rate equations that characterize the population of the qubit eigen-states, directly in the steady state, *by-passing* the transient dynamics. The expressions for these rates make transparent the cooling protocol that was explained in the main manuscript and also allow to estimate the fidelity of this protocol.

Given that the experimental parameters given in Table I show minor asymmetry between the two qubits/cavities/couplings, our approach is presented for a setup without asymmetry in the parameter values, *i.e.* $g_A = g_B = g$, $\Gamma_{1A} = \Gamma_{1B} = \Gamma_1$, and $\Gamma_{\phi A} = \Gamma_{\phi B} = \Gamma_\phi$. Furthermore, the experimental energy scales correspond to the hierarchy $\Delta^{\text{qc}} \gg g, J \gg \kappa_\pm \gg \Gamma_1 \gg \Gamma_\phi$. In order to make the calculation more transparent, we have made the approximation $\Delta_+^{\text{qc}} \approx \Delta_-^{\text{qc}} \equiv \Delta^{\text{qc}}$, where $\Delta_\pm^{\text{qc}} \equiv \omega^{\text{q}} - \omega_\pm^{\text{c}}$ is the qubit-cavity detuning. This is justified when $\Delta_\pm^{\text{qc}} \gg J$.

The non-linear light-matter coupling in $H_{\sigma,a}$ can be eliminated by means of a second-order perturbation theory in g/Δ (namely a Schrieffer-Wolff transformation [4]). This reveals a photon-mediated retarded interaction between the qubits, providing an experimental realization of a two-site transverse-field XY model. We furthermore decompose the photon fields into mean fields plus bosonic fluctuations: $A \equiv \bar{A} + D$, $a \equiv \bar{a} + d$, $\bar{N}_+ \equiv |\bar{A}|^2$ and $N_- \equiv |\bar{a}|^2$ where, to the lowest order in the light-matter coupling,

$$\bar{A} \simeq \frac{\sqrt{2}\epsilon^{\text{d}} \cos(\phi/2)}{-\Delta_+^{\text{cd}} + i\kappa_+/2} \text{ and } \bar{a} \simeq -\frac{i\sqrt{2}\epsilon^{\text{d}} \sin(\phi/2)}{-\Delta_-^{\text{cd}} + i\kappa_-/2}. \quad (8)$$

We neglect the resulting quadratic terms in the fluctuations which couple to the qubits, *e.g.* $(g/\Delta^{\text{qc}})^2 D^\dagger D \sigma_i^z$ [5]. Notice that for fixed drive amplitude ϵ^{d} at relevant drive frequencies $\omega^{\text{d}} < \omega_\pm^{\text{c}}$, the intra-cavity field is larger at the symmetric mode because the detuning is smaller. As a result, the value of ϕ at which $\bar{N}_+ = \bar{N}_-$ is not generically equal to $\pi \pm \pi/2$. Rather, it is given by $\tan \phi/2 = \pm \frac{\Delta_-^{\text{cd}}}{\Delta_+^{\text{cd}}}$. To maintain consistency with Schrieffer-Wolff perturbation theory we make sure to work in regimes where the photon fluctuations are relatively small [6]. Altogether, this yields the following Hamiltonian

$$\tilde{H} = \tilde{H}_\sigma + \tilde{H}_{\sigma,d} + \tilde{H}_d, \quad (9)$$

with

$$\tilde{H}_\sigma = \sum_i \mathbf{h} \cdot \frac{\boldsymbol{\sigma}_i}{2} - \frac{1}{2} J \left(\frac{g}{\Delta^{\text{qc}}} \right)^2 [\sigma_A^x \sigma_B^x + \sigma_A^y \sigma_B^y], \quad (10)$$

where

$$h_A^x = 2 \frac{g}{\Delta^{\text{qc}}} \epsilon^{\text{d}} \cos(\phi/2), \quad h_A^y = -2 \frac{g}{\Delta^{\text{qc}}} \epsilon^{\text{d}} \sin(\phi/2), \quad (11)$$

$$h_B^x = 2 \frac{g}{\Delta^{\text{qc}}} \epsilon^{\text{d}} \cos(\phi/2), \quad h_B^y = 2 \frac{g}{\Delta^{\text{qc}}} \epsilon^{\text{d}} \sin(\phi/2), \quad (12)$$

$$h_{A,B}^z = \tilde{\Delta}^{\text{qd}}, \quad (13)$$

$$\tilde{\Delta}^{\text{qd}} \equiv \Delta^{\text{qd}} + \Delta^{\text{qc}} (g/\Delta^{\text{qc}})^2 \left[1 + 2 (\epsilon^{\text{d}}/\Delta^{\text{cd}})^2 (1 + \Delta^{\text{cd}}/\Delta^{\text{qc}}) \right]. \quad (14)$$

Here, $\tilde{\Delta}^{\text{qd}}$ represents the detuning between the drive tone and the Stark-shifted qubit frequency $\tilde{\omega}^{\text{d}}$. Under the approximation that $\Delta_+^{\text{cd}} \approx \Delta_-^{\text{cd}}$ it is independent of phase; experimentally it has a slight phase dependence.

The eigenstates of $\tilde{H}_\sigma$ are, to first order in g/Δ^{qc} ,

$$|\tilde{T}^+\rangle = |T^+\rangle + \sqrt{2} \frac{\epsilon^{\text{d}}}{\tilde{\Delta}^{\text{qd}}} \frac{g}{\Delta^{\text{qc}}} [\cos(\phi/2)|T_0\rangle - i \sin(\phi/2)|S\rangle], \quad (15)$$

$$|\tilde{S}\rangle = |S\rangle - \sqrt{2} i \frac{\epsilon^{\text{d}}}{\tilde{\Delta}^{\text{qd}}} \frac{g}{\Delta^{\text{qc}}} \sin(\phi/2) [|T^+\rangle - |T^-\rangle], \quad (16)$$

$$|\tilde{T}_0\rangle = |T_0\rangle - \sqrt{2} \frac{\epsilon^{\text{d}}}{\tilde{\Delta}^{\text{qd}}} \frac{g}{\Delta^{\text{qc}}} \cos(\phi/2) [|T^+\rangle - |T^-\rangle], \quad (17)$$

$$|\tilde{T}^-\rangle = |T^-\rangle - \sqrt{2} \frac{\epsilon^{\text{d}}}{\tilde{\Delta}^{\text{qd}}} \frac{g}{\Delta^{\text{qc}}} [\cos(\phi/2)|T_0\rangle - i \sin(\phi/2)|S\rangle]. \quad (18)$$

The corresponding eigenenergies of $\tilde{H}_\sigma$ are

$$\begin{aligned} E_{\tilde{T}^+} &\simeq \tilde{\Delta}^{\text{qd}} + 2(n_+ \chi_+ + n_- \chi_-) \\ E_{\tilde{S}} &\simeq \delta, \\ E_{\tilde{T}_0} &\simeq -\delta, \\ E_{\tilde{T}^-} &\simeq -\tilde{\Delta}^{\text{qd}} - 2(n_+ \chi_+ + n_- \chi_-), \end{aligned} \quad (19)$$

where $\delta \equiv J(g/\Delta^{\text{qc}})^2$ and $n_\pm$ is the Fock state of the relevant coupled cavity mode. The degeneracy between $|S\rangle$ and $|T_0\rangle$ has been lifted by the effective photon-mediated qubit-qubit interaction. The photon fluctuations on top of the coherent mean-field part are governed by the quadratic Hamiltonian

$$\tilde{H}_d = \Delta_+^{\text{cd}} D^\dagger D + \Delta_-^{\text{cd}} d^\dagger d. \quad (20)$$

We shall assume that these non-interacting degrees of freedom equilibrate with the outside environment at zero temperature. Furthermore, these photon fluctuations couple to the qubits via

$$\begin{aligned} \tilde{H}_{\sigma,d} = \frac{1}{\sqrt{2}} \left(\frac{g}{\Delta^{\text{qc}}} \right)^2 \epsilon^{\text{d}} \left\{ (\Delta^{\text{qc}}/\Delta_+^{\text{cd}} + 1/2) [\cos(\phi/2)(\sigma_A^z + \sigma_B^z) - i \sin(\phi/2)(\sigma_A^z - \sigma_B^z)] D^\dagger \right. \\ \left. + (\Delta^{\text{qc}}/\Delta_-^{\text{cd}} + 1/2) [\cos(\phi/2)(\sigma_A^z - \sigma_B^z) - i \sin(\phi/2)(\sigma_A^z + \sigma_B^z)] d^\dagger \right\} + \text{h.c.} \end{aligned} \quad (21)$$

which induces transitions between the eigenstates of $\tilde{H}_\sigma$.

We integrate the photon-fluctuation degrees of freedom by treating $\tilde{H}_{\sigma,d}$ as a perturbation to $\tilde{H}_\sigma$ and we derive the dynamics of the reduced density matrix of the qubit sector, ρ_σ . Under standard assumptions, we obtain the following non-equilibrium steady-state Master equation [4]

$$\partial_t \rho_\sigma^{\text{NESS}} = 0 = -i \sum_k E_k [|k\rangle\langle k|, \rho_\sigma^{\text{NESS}}] + \sum_{kl} \Gamma_{k \rightarrow l} \mathcal{D}[|l\rangle\langle k|] \rho_\sigma^{\text{NESS}}, \quad (22)$$

where k and l span the eigenstates of $\tilde{H}_\sigma$.

The $\Gamma_{k \rightarrow l}$'s are steady-state transition rates between the qubit eigenstates. They read $\Gamma_{k \rightarrow l} = \Gamma_{k \rightarrow l}^b + \Gamma_{k \rightarrow l}^d$ with

$$\Gamma^b = \begin{pmatrix} - & \Gamma_1 & \Gamma_1 & 0 \\ 0 & - & \Gamma_\phi & \Gamma_1 \\ 0 & \Gamma_\phi & - & \Gamma_1 \\ 0 & 0 & 0 & - \end{pmatrix} \quad (23)$$

originating from the qubit decay and pure dephasing processes, and with the non-equilibrium drive part given by

$$\Gamma_{k \rightarrow l}^d = 2\pi |\Lambda_{kl}^+|^2 \rho_+(E_k - E_l + \omega^{\text{d}}) + 2\pi |\Lambda_{kl}^-|^2 \rho_-(E_k - E_l + \omega^{\text{d}}), \quad (24)$$

where

$$\Lambda^+ = \lambda_+ \begin{pmatrix} - & \alpha & \beta & 0 \\ \alpha & - & \sin(\phi/2) & \alpha \\ \beta & \sin(\phi/2) & - & \beta \\ 0 & \alpha & \beta & - \end{pmatrix}, \quad \Lambda^- = \lambda_- \begin{pmatrix} - & \beta & \alpha & 0 \\ \beta & - & \cos(\phi/2) & \beta \\ \alpha & \cos(\phi/2) & - & \alpha \\ 0 & \beta & \alpha & - \end{pmatrix}, \quad (25)$$

in which $\lambda_{\pm} \equiv \sqrt{2}(g/\Delta^{\text{qc}})^2 \epsilon^{\text{d}} (1/2 + \Delta^{\text{qc}}/\Delta_{\pm}^{\text{cd}})$, $\alpha \equiv \sqrt{2}(g/\Delta^{\text{qc}})(\epsilon^{\text{d}}/\tilde{\Delta}^{\text{qd}}) \cos \phi$ and $\beta \equiv \sqrt{2}(g/\Delta^{\text{qc}})(\epsilon^{\text{d}}/\tilde{\Delta}^{\text{qd}}) \sin \phi$. Matrix rows (columns) are ordered according to $|\tilde{T}_-\rangle$, $|\tilde{T}_0\rangle$, $|\tilde{S}\rangle$, and $|\tilde{T}_+\rangle$ from left to right (top to bottom). Note that in equilibrium, *i.e.* $\epsilon^{\text{d}} = 0$, one has $\Gamma_{k \rightarrow l}^{\text{d}} = 0$.

One can re-write the Master equation (22) as the rate equations governing the population of each of the eigenstates $|k\rangle$ of $\tilde{H}_{\sigma}$, $n_k \equiv \langle k | \rho_{\sigma} | k \rangle$:

$$\frac{dn_k^{\text{NESS}}}{dt} = 0 = \sum_l n_l^{\text{NESS}} \Gamma_{l \rightarrow k} - n_k^{\text{NESS}} \Gamma_{k \rightarrow l}, \quad (26)$$

which in conjugation with the conservation law $\sum_k n_k = 1$, provides a direct access to the non-equilibrium steady-state fidelities, by-passing the transient dynamics.

Protocol for generating entanglement

By judiciously tuning the drive frequency amplitude ϵ^{d} , frequency ω^{d} , and phase ϕ , one can drive the qubit subsystem to one of the maximally entangled eigen-states. We will discuss the protocol to achieve convergence to the singlet state, $\rho_{\sigma}^{\text{NESS}} \approx |\tilde{S}\rangle\langle\tilde{S}|$ and compute its fidelity $n_{\tilde{S}}^{\text{NESS}} \equiv \langle \tilde{S} | \rho_{\sigma}^{\text{NESS}} | \tilde{S} \rangle$. The extension to cooling to the triplet state is straightforward.

By selecting ω^{d} such that $\omega^{\text{d}} = \omega_-^{\text{c}} + E_{\tilde{S}} - E_{\tilde{T}_-}$, *i.e.* $\omega_{|\tilde{S}\rangle}^{\text{d}}(-) \approx \frac{1}{2} [\omega_-^{\text{c}} + \tilde{\omega}^{\text{q}} + 2\chi + \delta]$, the ρ^- term in the expression of $\Gamma_{\tilde{T}_- \rightarrow \tilde{S}}^{\text{d}}$ given in Eq. (24) is maximized to $2/(\pi\kappa_-)$. Therefore, we have a maximal pumping rate

$$\Gamma_{\tilde{T}_- \rightarrow \tilde{S}}^{\text{d}} \approx 400 \left(\frac{g}{\Delta^{\text{qc}}} \right)^6 \frac{|\epsilon^{\text{d}}|^4}{(\Delta^{\text{qc}})^2} \frac{\cos^2(\phi)}{\kappa_-}. \quad (27)$$

This pumping rate is labeled Γ_p in the main text. The mechanism associated with this rate corresponds to a two-photon process in which the incoming energy is used to add a photon in the antisymmetric cavity mode and simultaneously perform a qubit transition from $|\tilde{T}_-\rangle$ to $|\tilde{S}\rangle$ (Stokes scattering). Note that the phase at which $\Gamma_{\tilde{T}_- \rightarrow \tilde{S}}^{\text{d}}$ goes to zero when cooling via the antisymmetric cavity mode is exactly equal to $\pi/2$ in this derivation because we have approximated $\Delta_+^{\text{cd}} \approx \Delta_-^{\text{cd}}$. Experimentally, when this equality does not hold, the phase at which the transition rate goes to zero is given by Equation (7).

By selecting ω^{d} such that $\omega^{\text{d}} = \omega_+^{\text{c}} + E_{\tilde{S}} - E_{\tilde{T}_+}$, *i.e.* $\omega_{|\tilde{S}\rangle}^{\text{d}}(+) \approx \frac{1}{2} [\omega_+^{\text{c}} + \tilde{\omega}^{\text{q}} + 2\chi + \delta]$, it is the ρ^+ term in the expression of $\Gamma_{\tilde{T}_+ \rightarrow \tilde{S}}^{\text{d}}$ given in Eq. (24) which is maximized. Therefore, the pumping rate is given by,

$$\Gamma_{\tilde{T}_+ \rightarrow \tilde{S}}^{\text{d}} \approx 400 \left(\frac{g}{\Delta^{\text{qc}}} \right)^6 \frac{|\epsilon^{\text{d}}|^4}{(\Delta^{\text{qc}})^2} \frac{\sin^2(\phi)}{\kappa_+}. \quad (28)$$

Here, it is the symmetric-cavity mode which is used in the Stokes scattering process. Noting that this transition rate goes to zero at $\phi = 0$ and at $\phi = \pi$, we see that the parity selection rules described phenomenologically and observed in the main text have been reproduced.

In both cases, if the condition $\Gamma_{\tilde{T}_- \rightarrow \tilde{S}}^{\text{d}} \gg \Gamma_{\tilde{S} \rightarrow \tilde{T}_-}^{\text{d}} = \Gamma_1$ is met, this facilitates a rapid pumping from the ground state $|\tilde{T}_-\rangle$ to the singlet state $|\tilde{S}\rangle$ via Stokes (red-shifted) scattering. One must also avoid off-resonant pumping from $|\tilde{S}\rangle$ to the $|\tilde{T}_+\rangle$ state, labeled as $\Gamma_{\tilde{S} \rightarrow \tilde{T}_+}^{\text{d}}$ here and as Γ'_p in the main text. Altogether, the cooling to the singlet state is achieved with high fidelity whenever the hierarchy

$$\Gamma_{\tilde{S} \rightarrow \tilde{T}_+}^{\text{d}} \ll \Gamma_1 \ll \Gamma_{\tilde{T}_- \rightarrow \tilde{S}}^{\text{d}} \quad (29)$$

is obeyed where the off-resonant rate from $\tilde{S}$ to $\tilde{T}_+$ is given by $\Gamma_{\tilde{T}_- \rightarrow \tilde{S}}^{\text{d}} \approx 25(g/\Delta^{\text{qc}})^2(|\epsilon^{\text{d}}|^2/\Delta^{\text{qc}}J)^2\kappa_- \cos^2(\phi)$ or $\Gamma_{\tilde{T}_+ \rightarrow \tilde{S}}^{\text{d}} \approx 25(g/\Delta^{\text{qc}})^2(|\epsilon^{\text{d}}|^2/\Delta^{\text{qc}}J)^2\kappa_+ \sin^2(\phi)$ depending whether it is the antisymmetric or the symmetric cavity

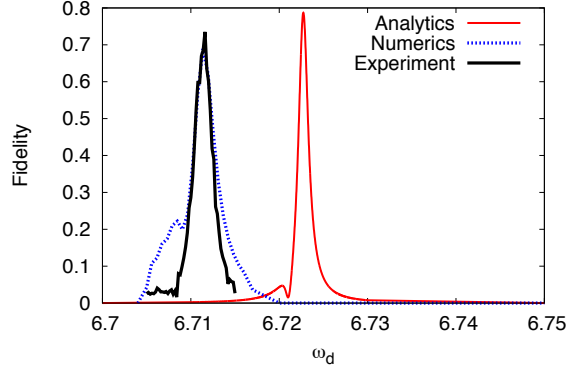


FIG. 6: Fidelity to singlet state (via the anti-symmetric mode) as a function of the drive frequency, ω_d . The near-perfect agreement between analytics, numerics [*i.e.* solving the full Master Equation (7)] and experimental methods is demonstrated. The small energy shift between the fidelity peaks of analytics and numerics/experiment can be mostly traced back to having neglected the correlated terms of the form $\Delta(g/\Delta)^2 D^\dagger D \sigma_i^z$ [see below Eq. (8)]. The parameters are summarized in Table 1 except for the Γ_ϕ 's that were reduced by 25 folds in the numerics and analytics, $\epsilon^d = 0.19 \times 2\pi$ GHz, and the relative phase between the two drives is taken to be $\phi = 0$.

mode which is used. The condition (29) transparently elucidates that the fidelity is the result of a delicate interplay between drive, cavity decay, qubit dissipation and light-matter coupling. In particular, ϵ^d enters both sides of the inequality above, and thus, large fidelity is obtained in a finite window of the drive strength ϵ^d .

In regimes where all the off-resonant rates are strongly suppressed compared to the other rates involved, *i.e.* $\Gamma_{k \rightarrow l}^d \approx 0$ except from $|k = \tilde{T}_-\rangle$ to $|l = \tilde{S}\rangle$, the steady-state population of the singlet state can be approximated to

$$n_{\tilde{S}}^{\text{NESS}} = \langle \tilde{S} | \rho^{\text{NESS}} | \tilde{S} \rangle \approx \frac{1}{1 + 2\Gamma_\phi/\Gamma_1} \frac{1 + \Gamma_\phi/\Gamma_1}{1 + \Gamma_1/\Gamma_{\tilde{T}_- \rightarrow \tilde{S}}^d}. \quad (30)$$

Therefore, if the hierarchy given in Eq. (29) is satisfied, the maximum steady-state fidelity to cooling to the singlet state simply reads

$$n_{\tilde{S}}^{\text{max}} \approx \frac{1}{1 + 2\Gamma_\phi/\Gamma_1}. \quad (31)$$

This result clearly highlights that, in contrast to the dephasing rate Γ_ϕ which is detrimental to the protocol, the dissipative qubit decay process is instrumental in achieving large fidelities.

Comparison with experiments

In Figure 6, we plot the steady-state fidelity of the protocol to stabilize the singlet state $|S\rangle$ via the antisymmetric cavity mode. Here the drive frequency ω_d is scanned, and for each value we plot the singlet occupation obtained from the experimental data, the full numerical results [*i.e.* solving the full Master Equation (7)], and the analytic results. The near-perfect agreement of the full numerics with the experimental data demonstrates the robustness of our modeling. The analytic results are also near-perfect except for a small energy shift of the optimal driving frequency which can be mostly traced back to having neglected the correlated terms of the form $\Delta^{\text{qc}}(g/\Delta^{\text{qc}})^2 D^\dagger D \sigma_i^z$ [see below Eq. (8)]. In Figure 7 we compare full numerics and experimental data for the transient dynamics towards the steady state. This further corroborates that the robustness of our modeling is not only confined to the steady-state.

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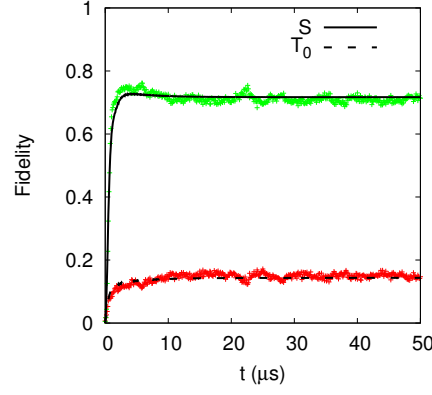


FIG. 7: Time dynamics of the singlet and triplet populations when driving to the singlet state via the symmetric mode. There is a near-perfect agreement between theory (*i.e.*, exact numerical treatment) and the experimental data. Our parameters are summarized in Table 1 except for the Γ_ϕ 's that were reduced by 25 folds in the numerics, $\omega^d = 6.7097 \times 2\pi$ GHz, $\epsilon^d = 0.145 \times 2\pi$ GHz, and the relative phase between the two drives is taken to be $\phi = \pi$.

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