

Supplementary Material: Cutoff-free Circuit Quantum Electrodynamics

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I. HEISENBERG EQUATIONS OF MOTION

In this section, we present the Heisenberg equations of motion in terms of flux variables [1, 2]. These equations were derived before by the authors [3] (see App. A), but the main steps are summarized below for clarity. The flux variable is defined at any node n in terms of the voltage at that node with respect to a fixed ground node

$$\Phi_n(t) \equiv \int_0^t dt' V_n(t'). \quad (1)$$

The classical Lagrangian is the sum of the Lagrangians for the Josephson junction, resonator, right and left waveguides, capacitive coupling between the resonator and the waveguides and the transmon-resonator capacitive coupling, respectively (let $U_j(\Phi_j)$ be the nonlinear Josephson potential):

$$\begin{aligned} \mathcal{L} = & \underbrace{\frac{1}{2} C_j \dot{\Phi}_j(t)^2 - U_j(\Phi_j(t))}_{\mathcal{L}_j} \\ & + \underbrace{\int_{0^+}^{L^-} dx \left[\frac{1}{2} c (\partial_t \Phi)^2 - \frac{1}{2l} (\partial_t \Phi)^2 \right]}_{\mathcal{L}_{\text{Res}}} \\ & + \underbrace{\int_{L^+}^{\infty} dx \left[\frac{1}{2} c (\partial_t \Phi_R)^2 - \frac{1}{2l} (\partial_x \Phi_R)^2 \right]}_{\mathcal{L}_{\text{RW}}} \\ & + \underbrace{\int_{-\infty}^{0^-} dx \left[\frac{1}{2} c (\partial_t \Phi_L)^2 - \frac{1}{2l} (\partial_x \Phi_L)^2 \right]}_{\mathcal{L}_{\text{LW}}} \\ & + \underbrace{\frac{1}{2} C_L [\dot{\Phi}_L(0^-, t) - \dot{\Phi}(0^+, t)]^2}_{\mathcal{L}_{C_L}} \\ & + \underbrace{\frac{1}{2} C_R [\dot{\Phi}_R(L^+, t) - \dot{\Phi}(L^-, t)]^2}_{\mathcal{L}_{C_R}} \\ & + \underbrace{\frac{1}{2} C_g [\dot{\Phi}_j(t) - \dot{\Phi}(x_0, t)]^2}_{\mathcal{L}_{C_g}}, \end{aligned} \quad (2)$$

From Eq. (2) one can derive, via a Legendre transformation followed by quantization [2, 4, 5], the Hamiltonian operator associated with the quantum circuit. The quantum Hamiltonian for $C_{R,L} \rightarrow 0$ is in Ref. 6. $C_{R,L} \neq 0$ leave equations of motion unchanged, but change boundary conditions (BCs) at $x = 0, L$. Importantly, Heisenberg equations of motion for the quantum flux operators

$\hat{\Phi}_j$, $\hat{\Phi}(x, t)$ and $\hat{\Phi}_{R,L}(x, t)$ turn out to be formally identical to Euler-Lagrange equations for (2) with classical fields promoted to operators.

To express the Heisenberg equations of motion in a compact way, we introduce the following notations. $\Phi_0 \equiv \frac{\hbar}{2e}$ is the superconducting flux quantum and E_j is the Josephson energy. $C_s \equiv C_g C_j / (C_g + C_j)$ is the series capacitance of C_j and C_g and $\gamma \equiv C_g / (C_g + C_j)$. There is a modified capacitance per unit length in the resonator due to the coupling to the transmon qubit at position x_0 :

$$c(x, x_0) \equiv c + C_s \delta(x - x_0). \quad (3)$$

c and l are the capacitance and inductance per unit length in the resonator and the waveguides.

We pass to unitless coordinates and operators ($v_p \equiv 1/\sqrt{lc}$)

$$\begin{aligned} x &\rightarrow \frac{x}{L}, \quad t \rightarrow \frac{t}{\frac{L}{v_p}}, \quad \omega \rightarrow \frac{\omega}{v_p} L, \\ \hat{\varphi} &\equiv 2\pi \frac{\hat{\Phi}}{\Phi_0}, \quad \hat{n} \equiv \frac{\hat{Q}}{2e} \end{aligned} \quad (4)$$

The newly introduced operators $\hat{\varphi}$ and $\hat{n}$ represent phase and number and are canonically conjugate: $[\hat{\varphi}_j, \hat{n}_j] = i$ and $[\hat{\varphi}(x, t), \hat{n}(x', t')] = i\delta(x - x')\delta(t - t')$. Below we use unitless capacitances $\chi_i \equiv C_i/(cL)$, $i = R, L, j, g, s$, and the unitless capacitance per unit length becomes

$$\chi(x, x_0) \equiv 1 + \chi_s \delta(x - x_0). \quad (5)$$

In terms of the quantities introduced, the Heisenberg equations of motion for the superconducting phase operators are:

$$\hat{\varphi}_j(t) + (1 - \gamma)\omega_j^2 \sin[\hat{\varphi}_j(t)] = \gamma \partial_t^2 \hat{\varphi}(x_0, t), \quad (6a)$$

$$[\partial_x^2 - \chi(x, x_0)\partial_t^2] \hat{\varphi}(x, t) = \chi_s \omega_j^2 \sin[\hat{\varphi}_j(t)] \delta(x - x_0), \quad (6b)$$

$$\partial_x^2 \hat{\varphi}_{R,L}(x, t) - \partial_t^2 \hat{\varphi}_{R,L}(x, t) = 0, \quad (6c)$$

with boundary conditions

$$\begin{aligned} -\partial_x \hat{\varphi}|_{x=1^-} &= -\partial_x \hat{\varphi}_R|_{x=1^+} \\ &= \chi_R \partial_t^2 [\hat{\varphi}(1^-, t) - \hat{\varphi}_R(1^+, t)], \end{aligned} \quad (7a)$$

$$\begin{aligned} -\partial_x \hat{\varphi}|_{x=0^+} &= -\partial_x \hat{\varphi}_L|_{x=0^-} \\ &= \chi_L \partial_t^2 [\hat{\varphi}_L(0^-, t) - \hat{\varphi}(0^+, t)], \end{aligned} \quad (7b)$$

$$\hat{\varphi}(x = x_0^-, t) = \hat{\varphi}(x = x_0^+, t), \quad (7c)$$

$$\begin{aligned} \partial_x \hat{\varphi}|_{x=x_0^+} - \partial_x \hat{\varphi}|_{x=x_0^-} - \chi_s \partial_t^2 \hat{\varphi}(x_0, t) \\ = \chi_s \omega_j^2 \sin[\varphi_j(t)]. \end{aligned} \quad (7d)$$

In Eqs. (6a) and (6b), the oscillation frequency is unitless $\omega_j^2 = 8\mathcal{E}_c\mathcal{E}_j$, in terms of unitless Josephson and charging energies

$$\mathcal{E}_{j,c} \equiv \sqrt{lc}L \frac{E_{j,c}}{\hbar}, \quad E_c \equiv \frac{e^2}{2C_j}. \quad (8)$$

Equations (6a-6b) are Eqs. (2-3) in the main text.

II. SPECTRAL REPRESENTATION OF THE GREEN'S FUNCTION

In this section we introduce a spectral representation of the Green's function. The Green's function enters the effective Heisenberg equation of motion for the superconducting phase of the transmon qubit (see Ref. 3 for a complete derivation). The resonator Green's function appears if one follows this aim in Eqs. (6a) and (6b): one has to solve for $\hat{\varphi}(x, t)$, which is driven by the qubit in Eq. (6b), and substitute into (6a). The resonator Green's function is defined as the response of the resonator fields, described by the left hand sides of Eqs. (6b-6c), to a δ -function source in space-time

$$[\partial_x^2 - \chi(x, x_0)\partial_t^2]G(x, t|x_0, t_0) = \delta(x - x_0)\delta(t - t_0), \quad (9)$$

obeying BCs (7a-7c) with $\hat{\varphi}(x, t)$ replaced by $G(x, t|x_0, t_0)$. Introducing Fourier transforms

$$\tilde{G}(x, x_0, \omega) = \int_{-\infty}^{\infty} dt G(x, t|x_0, t_0) e^{+i\omega(t-t_0)}, \quad (10a)$$

$$G(x, t|x_0, t_0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{G}(x, x_0, \omega) e^{-i\omega(t-t_0)}, \quad (10b)$$

Equation (9) becomes a Helmholtz equation

$$[\partial_x^2 + \omega^2\chi(x, x_0)]\tilde{G}(x, x_0, \omega) = \delta(x - x_0). \quad (11)$$

while the BCs (7a-7c) are transformed by replacing $\partial_t \rightarrow -i\omega$ to

$$\begin{aligned} \partial_x \tilde{G}|_{x=1^-} &= \partial_x \tilde{G}|_{x=1^+} \\ &= \chi_R \omega^2 \left(\tilde{G}|_{x=1^-} - \tilde{G}|_{x=1^+} \right), \end{aligned} \quad (12a)$$

$$\begin{aligned} \partial_x \tilde{G}|_{x=0^-} &= \partial_x \tilde{G}|_{x=0^+} \\ &= \chi_L \omega^2 \left(\tilde{G}|_{x=0^-} - \tilde{G}|_{x=0^+} \right). \end{aligned} \quad (12b)$$

$$\tilde{G}|_{x=x_0^+} = \tilde{G}|_{x=x_0^-}, \quad (12c)$$

$$\partial_x \tilde{G}|_{x=x_0^+} - \partial_x \tilde{G}|_{x=x_0^-} + \chi_s \omega^2 \tilde{G}|_{x=x_0} = 1, \quad (12d)$$

Lastly, outgoing BCs at infinity model the baths:

$$\partial_x \tilde{G}(x, x_0, \omega)|_{x \rightarrow \pm\infty} = \pm i\omega \tilde{G}(x \rightarrow \pm\infty, x_0, \omega). \quad (13)$$

Excitations leaving the resonator never reflect back towards it.

A. Spectral representation of Green's function for $\chi_{R,L} = 0$

Setting $\chi_R = \chi_L = 0$ (amounting to a closed resonator) imposes Neumann BC $\partial_x \tilde{G}|_{x=0,1} = 0$ and the problem for $\tilde{G}$ is Hermitian. $\tilde{G}$ can be expanded in terms of a discrete set of normal modes satisfying

$$\partial_x^2 \tilde{\varphi}_n(x) + \chi(x, x_0)\omega_n^2 \tilde{\varphi}_n(x) = 0, \quad (14a)$$

$$\partial_x \tilde{\varphi}_n(x)|_{x=0,1} = 0. \quad (14b)$$

An important feature of the modes is that their derivative is discontinuous

$$-\partial_x \tilde{\varphi}_n(x)|_{x_0^+}^{x_0^-} = \chi_s \omega_n^2 \tilde{\varphi}_n(x_0), \quad (15)$$

Physically, this is the continuity equation at x_0 , or current conservation. The mode amplitude at x_0 is suppressed. These observations lead us to name this set of resonator eigenmodes the *current-conserving (CC) basis*.

The CC basis eigenfrequencies obey a transcendental equation

$$\sin(\omega_n) + \chi_s \omega_n \cos(\omega_n x_0) \cos[\omega_n(1 - x_0)] = 0, \quad (16)$$

while the eigenfunctions read

$$\tilde{\varphi}_n(x) \propto \begin{cases} \cos[\omega_n(1 - x_0)] \cos(\omega_n x), & 0 < x < x_0 \\ \cos(\omega_n x_0) \cos[\omega_n(1 - x)], & x_0 < x < 1 \end{cases} \quad (17)$$

and the basis is orthonormal over $[0, 1]$:

$$\int_0^1 dx \chi(x, x_0) \tilde{\varphi}_m(x) \tilde{\varphi}_n(x) = \delta_{mn}. \quad (18)$$

Equation (16) can be solved numerically or asymptotically as $n \rightarrow \infty$, as we do in Sec. V.

The spectral representation of $\tilde{G}(x, x', \omega)$ [7-9] is

$$\tilde{G}(x, x', \omega) = \sum_{n \in \mathbb{N}} \frac{\tilde{\varphi}_n(x) \tilde{\varphi}_n(x')}{\omega^2 - \omega_n^2} = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{2\omega} \frac{\tilde{\varphi}_n(x) \tilde{\varphi}_n(x')}{\omega - \omega_n}, \quad (19)$$

since $\omega_{-n} = -\omega_n$ and $\tilde{\varphi}_{-n}(x) = \tilde{\varphi}_n(x)$.

B. Spectral representation of Green's function for $\chi_{R,L} \neq 0$

If the resonator is open, $\chi_{L,R} \neq 0$, we resort to a spectral representation in terms of a discrete set of non-Hermitian modes [3] that carry constant flux away from the resonator, Constant Flux (CF) modes [10]. CF modes satisfy the homogeneous wave equation

$$\partial_x^2 \tilde{\varphi}_n(x, \omega) + \chi(x, x_0) \omega_n^2(\omega) \tilde{\varphi}_n(x, \omega) = 0, \quad (20)$$

with BCs (12a)-(12c) and (13). Both the modes $\tilde{\varphi}_n(x, \omega)$ and their frequencies $\omega_n(\omega)$ depend on the source frequency ω .

An outgoing plane wave solution for the left and right waveguides obeying (13), is

$$\tilde{\varphi}_n(x, \omega) = \begin{cases} A_n^< e^{i\omega_n(\omega)x} + B_n^< e^{-i\omega_n(\omega)x}, & 0 < x < x_0 \\ A_n^> e^{i\omega_n(\omega)x} + B_n^> e^{-i\omega_n(\omega)x}, & x_0 < x < 1 \\ C_n e^{i\omega x}, & x > 1 \\ D_n e^{-i\omega x}, & x < 0 \end{cases} \quad (21)$$

Applying BCs (12c-12b) leads to a transcendental equation analogous to the closed case which fixes the parametric dependence $\omega_n(\omega)$ [3].

The CF modes satisfy now a biorthonormality [10] condition

$$\int_0^1 dx \chi(x, x_0) \tilde{\varphi}_m^*(x, \omega) \tilde{\varphi}_n(x, \omega) = \delta_{mn}, \quad (22)$$

where $\{\tilde{\varphi}_m(x, \omega)\}$ obey the Hermitian adjoint of (20). $\tilde{\varphi}_n(x, \omega)$ and $\tilde{\varphi}_n^*(x, \omega)$ are right and left eigenfunctions and obey $\tilde{\varphi}_n(x, \omega) = \tilde{\varphi}_n^*(x, \omega)$.

The CF mode spectral representation of the Green's function of the open resonator is

$$\tilde{G}(x, x', \omega) = \sum_n \frac{\tilde{\varphi}_n(x, \omega) \tilde{\varphi}_n^*(x', \omega)}{\omega^2 - \omega_n^2(\omega)}. \quad (23)$$

There are two sets of poles of $\tilde{G}(x, x', \omega)$ in the complex plane. When the denominator of (23) vanishes, $\omega = \omega_n(\omega)$, which corresponds to quasi-bound eigenfrequencies that obey

$$\begin{aligned} & [e^{2i\omega_n} - (1 - 2i\chi_L \omega_n)(1 - 2i\chi_R \omega_n)] \\ & + \frac{i}{2} \chi_s \omega_n [e^{2i\omega_n x_0} + (1 - 2i\chi_L \omega_n)] \\ & \times [e^{2i\omega_n(1-x_0)} + (1 - 2i\chi_R \omega_n)] = 0. \end{aligned} \quad (24)$$

The solutions reside in the lower half of complex ω -plane and come in symmetric pairs with respect to the $\Im\{\omega\}$ axis, i.e. if ω_n satisfies (24), so does $-\omega_n^*$. Therefore the eigenfrequencies are

$$\omega_n = \begin{cases} -i\kappa_0, & n = 0 \\ +\nu_n - i\kappa_n, & n \in +\mathbb{N} \\ -\nu_n - i\kappa_n, & n \in -\mathbb{N} \end{cases} \quad (25)$$

where $\nu_n > 0$ and $\kappa_n > 0$ are the oscillation frequency and decay rates of quasi-bound mode n , respectively. The dependence of κ_n on mode number n is plotted in Fig. 2 of the main letter. Note the existence of a pole at $\omega = 0$, which comes from the ω -dependence of CF states $\tilde{\varphi}_n(x, \omega)$ [10].

III. MULTIMODE JAYNES-CUMMINGS HAMILTONIAN

The classical Hamiltonian for the cQED system can be found from the circuit Lagrangian (2) [6]

$$\begin{aligned} \mathcal{H}_{sys} = & 4\mathcal{E}_c n_j^2(t) - \mathcal{E}_j \cos[\varphi_j(t)] \\ & + \int_0^1 dx \left\{ \frac{n^2(x, t)}{2\chi(x, x_0)} + \frac{1}{2} [\partial_x \varphi(x, t)]^2 \right\} \\ & + 2\pi\gamma z n_j(t) \int_0^1 dx \frac{n(x, t)}{\chi(x, x_0)} \delta(x - x_0), \end{aligned} \quad (26)$$

where $z \equiv Z/R_Q$ where $Z \equiv \sqrt{l/c}$ is the characteristic impedance of the resonator and $R_Q \equiv h/(2e)^2$ is the superconducting resistance quantum. The modification in capacitance per length originates from the system Lagrangian that contains the gauge-invariant qubit-resonator coupling $\chi_g[\dot{\varphi}_j(t) - \dot{\varphi}(x_0, t)]^2/2$. In contrast, a phenomenological product coupling $\chi_g \dot{\varphi}_j(t) \dot{\varphi}(x_0, t)$ would yield a $\mathcal{H}_{sys}$ with $\chi_s = 0$ which results in bare resonator modes.

For the purpose of quantizing $\mathcal{H}_{sys}$, we find the spectrum of the resonator by solving the corresponding Helmholtz eigenvalue problem that has been discussed in Sec. (II A). We find

$$\begin{aligned} \hat{\mathcal{H}}_{sys} \equiv & \frac{\omega_j}{4} \left\{ \hat{\mathcal{Y}}_j^2 - \frac{\sqrt{2}}{\epsilon} \cos[(2\epsilon^2)^{1/4} \hat{\mathcal{X}}_j] \right\} \\ & + \sum_n \left\{ \frac{\nu_n}{4} [\hat{\mathcal{X}}_n^2 + \hat{\mathcal{Y}}_n^2] + g_n \hat{\mathcal{Y}}_j \hat{\mathcal{Y}}_n \right\}, \end{aligned} \quad (27)$$

where we have defined the canonically conjugate variables $\hat{\mathcal{X}}_l \equiv (\hat{a}_l + \hat{a}_l^\dagger)$ and $\hat{\mathcal{Y}}_l \equiv -i(\hat{a}_l - \hat{a}_l^\dagger)$, where $\hat{a}_l$ represent the boson annihilation operator of sector $l \equiv j, c$. Moreover, $\omega_j \equiv \sqrt{8\mathcal{E}_j \mathcal{E}_c}$ and $\epsilon \equiv \sqrt{\mathcal{E}_c/\mathcal{E}_j}$ is a measure for the strength of transmon nonlinearity. For $\epsilon = 0$, we recover $\omega_j(\hat{\mathcal{X}}_j^2 + \hat{\mathcal{Y}}_j^2)/4$, the Hamiltonian of a simple harmonic oscillator. In the transmon regime where $\epsilon \ll 1$, the leading contribution is $-\sqrt{2}\epsilon\omega_j \hat{\mathcal{X}}_j^4/48$. The coupling between qubit and the n th CC mode of the resonator is

$$g_n = \frac{1}{2} \gamma \sqrt{\chi_j} \sqrt{\omega_j \nu_n} \tilde{\varphi}_n(x_0). \quad (28)$$

There are typically two approaches to diagonalize Eq. (27). In the first approach, assuming that the qubit nonlinearity is strong, one performs a two level reduction. Then, the multimode Rabi Hamiltonian can be derived from Eq. (27) by projecting the quadratures to

Pauli sigma matrices, $\hat{\mathcal{X}}_j \rightarrow \hat{\sigma}^x$ and $\hat{\mathcal{Y}}_j \rightarrow \hat{\sigma}^y$, which yields

$$\hat{\mathcal{H}}_{\text{Rabi}} = \frac{\omega_j}{2} \hat{\sigma}^z + \sum_n \nu_n \hat{a}_n^\dagger \hat{a}_n - \sum_n g_n (\hat{a}_n - \hat{a}_n^\dagger) (\hat{\sigma}^- - \hat{\sigma}^+). \quad (29)$$

In the rotating wave approximation, Eq. (29) transforms into the multimode Jaynes-Cummings Hamiltonian

$$\hat{\mathcal{H}}_{\text{JC}} = \frac{\omega_j}{2} \hat{\sigma}^z + \sum_n \nu_n \hat{a}_n^\dagger \hat{a}_n + \sum_n g_n (\hat{\sigma}^+ \hat{a}_n + \hat{\sigma}^- \hat{a}_n^\dagger) \quad (30)$$

used in the main text. Analytic results can be found for the Purcell decay rate and the Lamb shift in the dispersive limit where $g_n \ll |\omega_j - \omega_n|$ [11]. In a Lindblad calculation, resonator losses are included by a Bloch-Redfield approach through the Master equation for the reduced density matrix of the resonator and qubit degrees of freedom $\dot{\hat{\rho}} = -i[\hat{\mathcal{H}}_{\text{JC}}, \hat{\rho}] + \frac{\kappa_n}{2} (2\hat{a}_n \hat{\rho} \hat{a}_n^\dagger - \{\hat{\rho}, \hat{a}_n^\dagger \hat{a}_n\})$, where κ_n can be replaced from the solutions to Eq. (24). The second approach treats the nonlinearity as a weak perturbation and is explained in the next section.

IV. WEAKLY NONLINEAR TRANSMON

In this section we summarize the steps necessary to derive Eq. (10) of the main text. The full development of multi scale perturbation theory is in Ref. 3. By keeping the lowest order nonlinearity (Kerr terms which are quartic in the transmon quadrature), the Hamiltonian can be rewritten in a new basis that diagonalizes the quadratic part

$$\hat{\mathcal{H}}_{\text{sys}} \equiv \frac{\beta_j}{4} (\hat{\mathcal{X}}_j^2 + \hat{\mathcal{Y}}_j^2) + \sum_n \frac{\beta_n}{4} (\hat{\mathcal{X}}_n^2 + \hat{\mathcal{Y}}_n^2) - \frac{\varepsilon \omega_j}{8} \left(u_j \hat{\mathcal{X}}_j + \sum_n u_n \hat{\mathcal{X}}_n \right)^4, \quad (31)$$

where $\varepsilon \equiv \sqrt{2}\epsilon/6$, $\beta_{j,n}$ are hybridized frequencies and $u_{j,n}$ are hybridization coefficients: $\hat{\mathcal{X}}_j = u_j \hat{\mathcal{X}}_j + \sum_n u_n \hat{\mathcal{X}}_n$.

The Heisenberg equations of motion for quadratures become a set of quantum Duffing equations coupled via the quartic terms

$$\ddot{\hat{\mathcal{X}}}_l(t) + \beta_l^2 \left\{ \hat{\mathcal{X}}_l(t) - \varepsilon_l \left[u_j \hat{\mathcal{X}}_j(t) + \sum_n u_n \hat{\mathcal{X}}_n(t) \right]^3 \right\} = 0, \quad (32)$$

where $\varepsilon_l \equiv \frac{\omega_j}{\beta_l} u_l \varepsilon$ for $l \equiv j, n$. Up to lowest order in the perturbation [3], we find an operator valued correction of the linear theory qubit-like frequency β_j :

$$\hat{\beta}_j = \beta_j - \frac{\sqrt{2}\epsilon}{4} \omega_j \left[u_j^4 \hat{\mathcal{H}}_j(0) + \sum_n 2u_j^2 u_n^2 \hat{\mathcal{H}}_n(0) \right], \quad (33a)$$

and an analogous correction of the resonator like frequency β_n as

$$\hat{\beta}_n = \beta_n - \frac{\sqrt{2}\epsilon}{4} \omega_j \left[u_n^4 \hat{\mathcal{H}}_n(0) + 2u_n^2 u_j^2 \hat{\mathcal{H}}_j(0) + \sum_{m \neq n} 2u_n^2 u_m^2 \hat{\mathcal{H}}_m(0) \right], \quad (33b)$$

where $\hat{\mathcal{H}}_l(0) \equiv \frac{1}{4} [\hat{\mathcal{X}}_l^2(0) + \hat{\mathcal{Y}}_l^2(0)]$ for $l = j, n$. In the main text, Eq. (10), the bar notation is dropped. The lowest order MSPT solution for the qubit quadrature becomes, in terms of renormalized frequencies $\hat{\beta}_{j,n}$, [3]

$$\hat{\mathcal{X}}_j^{(0)}(t) = u_j \frac{\hat{a}_j(0) e^{-i\hat{\beta}_j t} + e^{-i\hat{\beta}_j t} \hat{a}_j(0)}{2 \cos\left(\frac{3\omega_j}{4} u_j^4 \varepsilon t\right)} + H.c. + \sum_n \left[u_n \frac{\hat{a}_n(0) e^{-i\hat{\beta}_n t} + e^{-i\hat{\beta}_n t} \hat{a}_n(0)}{2 \cos\left(\frac{3\omega_j}{4} u_n^4 \varepsilon t\right)} + H.c. \right]. \quad (34)$$

This equation takes into account corrections up to $\mathcal{O}(\varepsilon)$ in frequencies. To extract these corrections, we must evaluate the expectation value of Eq. (34) with respect to the initial density matrix. We chose $\hat{\rho} = |\Psi\rangle_j \langle\Psi|_j \otimes |0\rangle_{\text{ph}} \langle 0|_{\text{ph}}$ with $|\Psi\rangle_j = (|0\rangle_j + |1\rangle_j)/\sqrt{2}$. The correction to the transmon like frequency is obtained from the Fourier components of $\langle \hat{\mathcal{X}}_j(t) \rangle$. This is the correction plotted in Fig. 3 of the main text.

V. ASYMPTOTIC BEHAVIOR OF LIGHT-MATTER COUPLING

In this section we find the asymptotic behavior of the eigenfrequencies ω_n and eigenmodes $\tilde{\varphi}_n(x)$ of the resonator discussed in the main text. This provides an analytical understanding of the high frequency suppression in the light-matter coupling g_n .

To point out the origin of the suppression that arise from a nonzero χ_s , let us consider the closed resonator ($\chi_{R,L} = 0$) case. Consider the special case of $x_0 = 0^+$ first. This is of experimental interest in order to achieve the maximum coupling to all modes of a resonator. Then, the transcendental Eq. (16) simplifies to

$$\sin(\omega_n) + \chi_s \omega_n \cos(\omega_n) = 0, \quad (35)$$

which can be rewritten as

$$\tan(\omega_n) = -\chi_s \omega_n. \quad (36)$$

The large ω_n solution for $\chi_s \neq 0$ is then obtained

$$\lim_{n \rightarrow \infty} \omega_n = n\pi - \frac{\pi}{2}, \quad (37)$$

which is independent of the value for χ_s . This implies that the effect of a nonzero χ_s on ω_n is a total shift $\pi/2$ (half of the free spectral range) in comparison with the case $\chi_s = 0$. Substituting $x_0 = 0^+$ in Eq. (17), the normalization factor $\mathcal{N}_n$ is found via Eq. (18) as

$$\int_0^1 dx \cos^2[\omega_n(1-x)] + \chi_s \cos^2(\omega_n) = \frac{1}{\mathcal{N}_n^2}, \quad (38)$$

which gives

$$\mathcal{N}_n = \frac{\sqrt{2}}{\sqrt{1 + \chi_s \cos^2(\omega_n)}}. \quad (39)$$

Therefore the eigenmode is found as

$$\tilde{\varphi}_n(x_0 = 0^+) = \frac{\sqrt{2} \cos(\omega_n)}{\sqrt{1 + \chi_s \cos^2(\omega_n)}}. \quad (40)$$

Using the trigonometric identity

$$\cos^2(\omega_n) = \frac{1}{1 + \tan^2(\omega_n)} \quad (41)$$

and Eq. (36) we can rewrite Eq. (40) as

$$\tilde{\varphi}_n(x_0 = 0^+) = \frac{\sqrt{2}}{\sqrt{1 + \chi_s + \chi_s^2 \omega_n^2}}, \quad (42)$$

which now provides the algebraic dependence of $\tilde{\varphi}_n(x_0)$ on ω_n . According to Eq. (42), for large enough ω_n ($\chi_s \omega_n \gg 1 + \chi_s$), we find

$$\tilde{\varphi}_n(x_0) \sim \frac{1}{\omega_n}, \quad (43)$$

where the symbol $\sim$ represents asymptotic equivalence. This imposes a natural cut-off on the light matter coupling for $n \rightarrow \infty$, since

$$g_n \propto \sqrt{\omega_n} \tilde{\varphi}_n(x_0) \sim \frac{1}{\sqrt{\omega_n}}. \quad (44)$$

Next, we would like to find the asymptotic behavior of ω_n and $\tilde{\varphi}_n(x_0)$ for a general x_0 . In order to bring Eq. (16) into a similar form to Eq. (36), we first replace $\sin(\omega_n) = \sin[\omega_n x_0 + \omega_n(1-x_0)]$ and then divide by $\cos(\omega_n x_0) \cos[\omega_n(1-x_0)]$ to obtain

$$\tan(\omega_n x_0) + \tan[\omega_n(1-x_0)] = -\chi_s \omega_n. \quad (45)$$

Next, the normalization factor $\mathcal{N}_n$ is found from Eq. (18) as

$$\mathcal{N}_n = \frac{\sqrt{2}}{\sqrt{x_0 \cos^2[\omega_n(1-x_0)] + (1-x_0) \cos^2(\omega_n x_0) + \chi_s \cos^2[\omega_n(1-x_0)] \cos^2(\omega_n x_0)}}, \quad (46)$$

Plugging this into Eq. (17) we find

$$\tilde{\varphi}_n(x_0) = \frac{\sqrt{2}}{\sqrt{1 + \chi_s + x_0 \tan^2(\omega_n x_0) + (1-x_0) \tan^2[\omega_n(1-x_0)]}} \quad (47)$$

Equations (45) and (47) provide the asymptotic behavior of ω_n , $\tilde{\varphi}_n(x_0)$ and g_n for a general x_0 .

VI. CHARACTERISTIC FUNCTION $D_j(s)$ AND ITS CONVERGENCE

In this section we derive the expression for the characteristic function $D_j(s)$ and compare its convergence in number of resonator modes with and without the modification we found for g_n .

Consider the Heisenberg-Langevin equations of motion corresponding to Hamiltonian (27) in the linear regime

($\epsilon = 0$) for $\hat{\mathcal{X}}_{j,n}(t)$ as

$$(d_t^2 + \omega_j^2) \hat{\mathcal{X}}_j(t) = - \sum_n 2g_n \omega_n \hat{\mathcal{X}}_n(t), \quad (48a)$$

$$(d_t^2 + 2\kappa_n d_t + \omega_n^2) \hat{\mathcal{X}}_n(t) = -2g_n \omega_j \hat{\mathcal{X}}_j(t) - \hat{f}_n(t), \quad (48b)$$

where κ_n and $\hat{f}_n$ are the decay rate and noise operator coming from coupling to the waveguide degrees of freedom [12].

Equations (48a-48b) are linear constant coefficient ODEs and can be solved exactly via the unilateral Laplace transform

$$\tilde{h}(s) = \int_0^\infty dt h(t) e^{-st}. \quad (49)$$

Taking the Laplace transform of Eqs. (48a-48b) we obtain

$$(s^2 + \omega_j^2) \hat{\mathcal{X}}_j(s) + \sum_n 2g_n \omega_n \hat{\mathcal{X}}_n(s) = s\hat{\mathcal{X}}_j(0) + \hat{\mathcal{X}}_j(0), \quad (50a)$$

$$(s^2 + 2\kappa_n s + \omega_n^2) \hat{\mathcal{X}}_n(s) + 2g_n \omega_j \hat{\mathcal{X}}_j(s) = (s + 2\kappa_n) \hat{\mathcal{X}}_n(0) + \hat{\mathcal{X}}_n(0) + \hat{f}_n(s). \quad (50b)$$

The solution for $\hat{\mathcal{X}}_j(s)$ then reads

$$\hat{\mathcal{X}}_j(s) = \frac{\hat{N}_j(s)}{D_j(s)}, \quad (51)$$

where the numerator

$$\begin{aligned} \hat{N}_j(s) &= s\hat{\mathcal{X}}_j(0) + \hat{\mathcal{X}}_j(0) \\ &- \sum_n \frac{2g_n \omega_n [(s + 2\kappa_n) \hat{\mathcal{X}}_n(0) + \hat{\mathcal{X}}_n(0) - \hat{f}_n(s)]}{s^2 + 2\kappa_n s + \omega_n^2}, \end{aligned} \quad (52)$$

contains the operator initial conditions and the denominator

$$D_j(s) \equiv s^2 + \omega_j^2 - \sum_n \frac{4g_n^2 \omega_j \omega_n}{s^2 + 2\kappa_n s + \omega_n^2}. \quad (53)$$

is the characteristic function whose roots give the hybridized poles of the full system. Therefore, we can represent $D_j(s)$ as

$$D_j(s) = (s - p_j)(s - p_j^*) \prod_n \frac{(s - p_n)(s - p_n^*)}{(s - z_n)(s - z_n^*)}, \quad (54)$$

where $p_{j,n} \equiv -\alpha_{j,n} - i\beta_{j,n}$ stand for the transmon-like and the n th resonator-like poles, respectively. Furthermore, $z_n \equiv -\kappa_n - i\sqrt{\omega_n^2 - \kappa_n^2}$ is the n th bare non-Hermitian resonator mode. The notation (p for poles and z for zeros) is chosen based on $1/D_j(s)$ that appears in the Laplace solution (51).

In order to compute the hybridized poles in practice, we need to truncate the number of resonator modes in $D_j(s)$. This truncation is only justified if the function $D_j(s)$ converges as we include more and more modes. First, note that without the correction give by χ_s this sum is divergent, since $g_n \sim \sqrt{\omega_n} \sim \sqrt{n}$ and for a fixed s we obtain

$$\frac{4g_n^2 \omega_j \omega_n}{s^2 + 2\kappa_n s + \omega_n^2} \sim \frac{\omega_n^2}{\omega_n^2} \sim 1. \quad (55)$$

Hence, the series is divergent. On the other hand, we found that for a non-zero χ_s , $g_n \sim 1/\sqrt{\omega_n} \sim 1/\sqrt{n}$. Therefore we find

$$\frac{4g_n^2 \omega_j \omega_n}{s^2 + 2\kappa_n s + \omega_n^2} \sim \frac{1}{\omega_n^2} \sim \frac{1}{n^2}, \quad (56)$$

and the series becomes convergent. In writing Eq. (56), we used the fact that $\omega_n \sim n$ and κ_n has a sublinear asymptotic behavior found numerically.

VII. DIVERGENCE IN THE WIGNER-WEISSKOPF THEORY OF SPONTANEOUS EMISSION

Divergence of the Purcell decay rate appears in other frameworks besides the dispersive limit Jaynes-Cummings model as well. In this appendix, we show that the spontaneous decay rate of a qubit coupled to continuum of modes is also *divergent*, unless the gauge invariance of the interaction is incorporated as presented in this manuscript. The impression of an (erroneous) finite decay rate in free space goes back to Wigner and Weisskopf's original work on spontaneous atomic decay, which implicitly makes a Markov approximation (See Sec. 6.3 of [13]). We emphasize that employing the Markov approximation always yields a finite value for the decay rate regardless of the form of spectral function for electromagnetic background.

To see this explicitly, we go over the Wigner-Weisskopf theory of spontaneous emission for a two-level system coupled to a continuum of modes inside an infinitely long 1D medium. In interaction picture, the Hamiltonian reads

$$\hat{\mathcal{H}}_I = \sum_k \hbar \left[g_k^*(x_0) \hat{\sigma}^+ \hat{a}_k e^{i(\omega_j - \omega_k)t} + H.c. \right], \quad (57)$$

which conserves the total number of excitations

$$\hat{N} \equiv \hat{\sigma}^+ \hat{\sigma}^- + \sum_{\vec{k}} \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}}. \quad (58)$$

As a result, a number conserving Ansatz for the wavefunction can be written as

$$|\Psi(t)\rangle = c_e(t) |e, 0\rangle + \sum_k c_{g,k}(t) |g, 1_k\rangle, \quad (59)$$

where there is either no photon in the cavity and the qubit is in excited state $|e\rangle$, or there is a photon at frequency ω_k with qubit in the ground state $|g\rangle$. By solving the Schrodinger equation we obtain the time evolution of the unknown probability amplitudes $c_e(t)$ and $c_{g,k}(t)$. Combining these equations yields an effective equation for $c_e(t)$ as

$$\dot{c}_e(t) = - \int_0^t dt' \mathcal{K}(t - t') c_e(t'), \quad (60)$$

where the memory Kernel $\mathcal{K}(\tau)$ is given by

$$\mathcal{K}(\tau) \equiv \sum_k |g_k(x_0)|^2 e^{i(\omega_j - \omega_k)\tau}. \quad (61)$$

Next, we replace the expression for $g_k(x_0)$, derived in Sec. III, as

$$|g_k(x_0)|^2 = \frac{\gamma \chi_s}{4} \omega_j \omega_k |\tilde{\varphi}_k(x_0)|^2. \quad (62)$$

Note that without respecting the gauge symmetry of interaction $|\tilde{\varphi}_k(x_0)| = \mathcal{N}(x_0)$ is k -independent. Moreover, the sum over k can be replaced as

$$\sum_k \rightarrow \frac{L}{2\pi} \int_0^\infty dk = \frac{L}{2\pi v_p} \int_0^\infty d\omega_k, \quad (63)$$

for a continuum of modes, where v_p is the phase velocity of the medium. Inserting Eqs. (62) and (63) into the effective Eq. (60) we obtain

$$\begin{aligned} \dot{c}_e(t) = & -\frac{1}{2\pi} \frac{\gamma \chi_s \omega_j \mathcal{N}^2(x_0) L}{4v_p} \\ & \times \int_0^\infty d\omega_k \omega_k \int_0^t dt' e^{i(\omega_j - \omega_k)(t-t')} c_e(t') \end{aligned} \quad (64)$$

Importantly, the integral over ω_k in Eq. (64) does not converge since the integrand grows unbounded as $\omega_k \rightarrow \infty$. To resolve this, Wigner and Weisskopf assumed that the dominant contribution comes from those modes of continuum whose frequency are close to the qubit frequency. Therefore, the factor ω_k can be replaced by ω_j and by extending the lower limit of integral over ω_k to $-\infty$ we can use the identity

$$\int_{-\infty}^{+\infty} d\omega_k e^{i(\omega_j - \omega_k)(t-t')} = 2\pi \delta(t-t'), \quad (65)$$

to arrive at a *finite* value for the spontaneous decay as

$$\dot{c}_e(t) \approx -\frac{\Gamma_{sp}}{2} c_e(t), \quad (66a)$$

$$\Gamma_{sp} \equiv \frac{\gamma \chi_s \omega_j^2 \mathcal{N}^2(x_0) L}{2v_p}. \quad (66b)$$

It is worth mentioning that using Markov approximation, one always obtains a finite expression for the spontaneous decay rate regardless of the form for the spectral function.

This happens because instead of integrating over the entire frequency span, the Markov approximation picks a small window around qubit frequency.

Next, we show how our natural high frequency cut-off for light-matter coupling resolves the divergence of Wigner-Weisskopf theory. First, note that applying Markov approximation is indeed unnecessary, since the Volterra Eq. (60) with the memory kernel

$$\mathcal{K}(\tau) = \frac{1}{2\pi} \frac{\gamma \chi_s \omega_j L}{4v_p} \int_0^\infty d\omega_k \omega_k |\tilde{\varphi}_k(x_0)|^2 e^{i(\omega_j - \omega_k)\tau}, \quad (67)$$

has an exact solution in Laplace domain as

$$\tilde{c}_e(s) = \frac{c_e(0)}{s + \tilde{\mathcal{K}}(s)}, \quad (68)$$

where $\tilde{\mathcal{K}}(s) \equiv \int_0^\infty d\tau \mathcal{K}(\tau) e^{-s\tau}$ is the Laplace transform and is found as

$$\tilde{\mathcal{K}}(s) = \frac{1}{2\pi} \frac{\gamma \chi_s \omega_j L}{4v_p} \int_0^\infty d\omega_k \frac{\omega_k |\tilde{\varphi}_k(x_0)|^2}{s + i(\omega_k - \omega_j)}. \quad (69)$$

Second, when the gauge-invariance of the interaction is incorporated, the mode amplitude is frequency dependent that experiences a high frequency suppression as

$$|\tilde{\varphi}_k(x_0)| \sim \frac{1}{\omega_k}. \quad (70)$$

Replacing Eq. (70) into expression (69) for $\tilde{\mathcal{K}}(s)$ we obtain

$$\tilde{\mathcal{K}}(s) \propto \int d\omega_k \frac{1}{\omega_k [s + i(\omega_k - \omega_j)]}. \quad (71)$$

Interestingly, with the corrected expression for the eigenmodes, the integrand behaves like $1/\omega_k^2$ at $\omega_k \rightarrow \infty$, and as a result the integral converges. Otherwise, the integrand behaves like a constant at $\omega_k \rightarrow \infty$ and the result is divergent.

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