

Supplementary Material for "Systematic time-coarse graining for driven quantum systems"

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Appendix A: Software framework

STCG is general and analytic, but it requires extensive symbolic manipulation since the number of terms grows rapidly with the order of approximation, making manual calculation of the TCG Lindbladian inconvenient at best and infeasible at worst. To this end, we developed ‘**QuantumGraining.jl**’, a Julia package that automates the TCG process efficiently for any order, calculating a symbolic expression for the effective Liouvillian in the form of a Hamiltonian and generalized Lindblad operators. We opt for **Julia** [1] due to its high performance, robust symbolic packages, and thriving scientific community. Our package can be used directly to aid analytic methods and pen-and-paper calculations, and can also be seamlessly integrated with other software packages such as *QuantumCumulants.jl* [2] or *QuantumOptics.jl* [3] for efficient numerical computation.

At the core of the STCG method is the calculation of the contraction coefficients $C_{l,r}$ defined in the previous section. Remarkably, this can be done independently from the operator algebra of the system, and is therefore fully encoded in a separate TCG process. The operator algebra is used for expressing the original Hamiltonian in a simple way, and more importantly, for writing the final Hamiltonian and pseudo-dissipators. In the current version of the code, the operator algebra is handled using the *QuantumCumulants.jl* package, while other symbolic functions are handled using the *Symbolics.jl* package. The details of calculating the STCG contributions, such as calculating singular contributions and enumerating the contributing diagrams, are detailed in App. D 1 and D 2.

Taking the Rabi model as an example, we first define the Hilbert space and the original model Hamiltonian as,

```
1 @variables g wc wa
2 Ω = [-wc - wa, wc + wa, -wc + wa, wc - wa]
3 gvec = (g/2).*[1, 1, 1, 1]
4
5 # Hilbert space definitions (QuantumCumulants.jl)
6 h_cav = FockSpace(:cavity)
7 h_atom = NLevelSpace(:atom, (:g,:e))
8 h = tensor(h_cav, h_atom)
9
10 # Operator definitions
11 @qnumbers a::Destroy(h) σ::Transition(h)
12 σm = σ(:e, :g); σp = σ(:g, :e)
13 hvec = [a*σm, a'*σp, a*σp, a'*σm]
```

With the Hamiltonian defined, we proceed to calculate the TCG Hamiltonian and dissipators,

```
1 g_eff, Ω_eff = effective_hamiltonian(hvec, gvec, Ω, order=2; as_dict=true)
2 γ_eff, Ω_eff = effective_dissipator(hvec, gvec, Ω, order=2)
```

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Behind the scenes, these two lines of code calculate all relevant contractions and automatically sum up their contributions. The objects returned are dictionaries, with the new operators as keys, and coupling strengths and frequencies as values. The results can either be printed out, for aiding analytical calculations, or converted to other packages for direct numerical calculation.

For example, for integrating with *QuantumOptics.jl*, first define a new set of compatible operators,

```

1  # Units
2  μs = 1; MHz = 1/μs
3
4  ### QuantumOptics.jl definitions
5  ha_qo = SpinBasis(1/2); hc_qo = FockBasis(100);
6  h_qo = hc_qo ⊗ ha_qo
7
8  # Operator definitions
9  σp_qo = sigmap(ha_qo); σm_qo = sigmam(ha_qo)
10 a_qo = destroy(hc_qo)
11 I_a = identityoperator(ha_qo); I_c = identityoperator(hc_qo)
12
13 p_sym = [g, ωc, ωa]
```

With the operators defined, we can generate a ‘QuantumOptics.jl’ object Hamiltonian and solve directly.

```

1  tspan = [0:0.01:120μs;]
2  ψ0 = coherentstate(hc_qo, 4.5) ⊗ spinup(ha_qo)
3
4  base_qc = [a, a', σm, σp, σ(:e, :e)]
5  Id = [I_c, I_a]
6  base_qo = [a_qo, a_qo', σm_qo, σp_qo, σp_qo*σm_qo, Id...]
7
8  H = hamiltonian_function(g_eff, Ω_eff, base_qc, base_qo, p_sym)
9
10 args = [2π*0.2MHz, 2π*2MHz, 2π*2.1MHz, 0.2μs]
11 tout, ψt = timeevolution.schroedinger_dynamic(tspan, ψ0, (t, ψ) -> H(t, ψ;
12 args=args));
13
14 plot(tout, real(expect(1, a_qo'*a_qo, ψt)), lw=2.5, label="a'*a - 1");
15
16 xlabel!("Time [μs]")
17 ylabel!("L<n>")
```

A full example notebook, including integration with ‘QuantumCumulants.jl’ and ‘QuantumOptics.jl’ is available on the Github repository [4].

Appendix B: TCG master equation derivation

As we discuss in the main text, we want to derive a closed formula for the TCG master equation. We start by expanding the unitary evolution in the following Dyson series:

$$\hat{U}(t) \equiv \sum_{k=0}^{\infty} \hat{U}_k(t) = \hat{I} + \sum_{k=1}^{\infty} (-i)^k \int_0^t \cdots \int_0^{t_{k-1}} dt_1 \cdots dt_k \hat{H}(t_1) \cdots \hat{H}(t_k) \quad (\text{B1})$$

The TCG procedure represents a low-pass filter of the system dynamics in the interaction picture can then be implemented by working exclusively with the time-averaged density matrix,

$$\bar{\rho}(t) = \overline{U(t)\rho_0 U^\dagger(t)} = \sum_{k=0}^{\infty} \sum_{l=0}^k \overline{\hat{U}_l \rho_0 \hat{U}_{k-l}^\dagger}(t) \quad (\text{B2})$$

where the time average of a time-dependent operator $\hat{O}(t)$ is defined as

$$\bar{\hat{O}}(t) := \int_{-\infty}^{\infty} dt' \tilde{f}_\tau(t-t') \hat{O}(t') \quad (\text{B3})$$

for some window function $\tilde{f}_\tau(t)$ normalized so that $\int_{-\infty}^{\infty} \tilde{f}_\tau(t) dt = 1$; the coarse-graining time scale τ , on the other hand, defines the shape of the filter. In what follows, we will denote its frequency-domain representation by $f_\tau(\omega)$.

Starting from Eq.(B2), we would like to obtain the generator of the time evolution of $\bar{\rho}(t)$. This requires finding at least an approximate inverse of the TCG operation, and we show in the following paragraphs that it can be done by perturbatively reversing Eq.(B2). Along this line of thought, the first thing to notice from the form of Eq.(B2) is that the time evolution of $\bar{\rho}(t)$ cannot be generated by a Hamiltonian in general, as discussed in the main text. Consequently, the correct ansatz is a general master equation generated by a quantum Liouvillian [5–8],

$$i \frac{\partial}{\partial t} \bar{\rho}(t) = \mathcal{L} \bar{\rho}(t) = \sum_{k=1}^{\infty} \mathcal{L}_k(t) \bar{\rho}(t). \quad (\text{B4})$$

Let us define the time-averaged state,

$$\bar{\rho}(t) = \sum_{k=0}^{\infty} \sum_{l=0}^k \overline{\hat{U}_l \rho \hat{U}_{k-l}^\dagger}(t) \equiv \sum_k \mathcal{E}_k(t) \rho \quad (\text{B5})$$

where we denote the time-averaging operation $\mathcal{E}(t) = \sum_k \mathcal{E}_k(t)$. By collecting powers of k we find,

$$\mathcal{E}_k(t) \rho = \sum_{l=0}^k \overline{\hat{U}_l \rho \hat{U}_{k-l}^\dagger}(t) \quad (\text{B6})$$

Using the definition of the time-average in Eq.(B3) and taking the derivative of $\bar{\rho}(t)$, the LHS of Eq.(B4) becomes

$$\begin{aligned} i \partial_t \bar{\rho}(t) &= \sum_{k=1}^{\infty} \sum_{l=1}^k \left(\overline{\hat{H} \hat{U}_{l-1} \rho \hat{U}_{k-l}^\dagger}(t) - \overline{\hat{U}_{l-1} \rho \hat{U}_{k-l}^\dagger \hat{H}}(t) \right) \\ &= \sum_{j=1}^{\infty} \mathcal{L}_j(t) \bar{\rho}(t) \\ &= \sum_{j=1}^{\infty} \mathcal{L}_j(t) \sum_{k=0}^{\infty} \mathcal{E}_k(t) \rho = \sum_{j=1}^{\infty} \mathcal{L}_j(t) \sum_{k-j=0}^{\infty} \mathcal{E}_{k-j}(t) \rho \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^k \mathcal{L}_j(t) \mathcal{E}_{k-j}(t) \rho \end{aligned} \quad (\text{B7})$$

Matching terms at order k and expanding $\mathcal{E}_{k-j}(t)$, we find the following recurrence relation for the Liouvillian expansion

$$\mathcal{L}_k(t) \rho = \sum_{l=1}^k \left(\overline{\hat{H} \hat{U}_{l-1} \rho \hat{U}_{k-l}^\dagger}(t) - \overline{\hat{U}_{l-1} \rho \hat{U}_{k-l}^\dagger \hat{H}}(t) \right) - \sum_{k=1}^{\infty} \sum_{j=1}^k \mathcal{L}_j(t) \mathcal{E}_{k-j}(t) \rho. \quad (\text{B8})$$

where we define $\mathcal{L}_0 \bar{\rho} \equiv 0$ for any density matrix $\bar{\rho}$. This formula relates $\mathcal{L}_k$ to all partial Liouvillians of order $r < k$.

Appendix C: Derivation of the contraction coefficients

1. The contraction super-operators

In the main text, we derive

$$\begin{aligned} \mathcal{L}_k(t)\bar{\rho} = & \sum_{l=1}^k \left(\overline{\hat{H}\hat{U}_{l-1}\bar{\rho}\hat{U}_{k-l}^\dagger}(t) - \overline{\hat{U}_{l-1}\bar{\rho}\hat{U}_{k-l}^\dagger\hat{H}}(t) \right) \\ & - \sum_{k'=0}^{k-1} \mathcal{L}_{k'}(t) \sum_{l=0}^{k-k'} \overline{\hat{U}_l\bar{\rho}\hat{U}_{k-k'-l}^\dagger} \end{aligned} \quad (\text{C1})$$

Due to their complicated time dependence, directly calculating the averages in Eq.(C1) is difficult. For example, even with only two operators we would have,

$$\overline{\hat{H}\hat{U}_1}(t) = \sum_{l_1, l_2} \frac{f_\tau(\omega_{l_1} + \omega_{l_2})e^{-i(\omega_{l_1} + \omega_{l_2})t} - f_\tau(\omega_{l_2})e^{-i\omega_{l_2}t}}{\omega_{l_1}} \hat{h}_{l_2} \hat{h}_{l_1}$$

which contains two terms at different frequencies. However, Eq.(C1) can be used to show that such a time average is always accompanied by the following product of time averages,

$$\overline{\hat{H}(t)\hat{U}_1(t)} = \sum_{l_1, l_2} \frac{f_\tau(\omega_{l_1})f_\tau(\omega_{l_2})e^{-i(\omega_{l_1} + \omega_{l_2})t} - f_\tau(\omega_{l_2})e^{-i\omega_{l_2}t}}{\omega_{l_1}} \hat{h}_{l_2} \hat{h}_{l_1}.$$

In previous work by Lee et al.[9], it has been shown that the time-coarse grained dynamics to the second-order are generated by two-point “contractions” which happen to be equivalent to covariances

$$\overline{PQ} = \overline{PQ} - \overline{P} \cdot \overline{Q} \quad (\text{C2a})$$

$$\overline{P\bar{\rho}Q} = \overline{P\bar{\rho}Q} - \overline{P} \cdot \bar{\rho} \cdot \overline{Q}. \quad (\text{C2b})$$

Note that by plugging in the time-averaged terms into the contraction, the terms at frequency ω_{l_2} always cancel each other out, and we have the following expression where the time dependence is homogeneous for each combination of $\hat{h}_{l_1}$ and $\hat{h}_{l_2}$:

$$\overline{HU}_1 = \sum_{l_1, l_2} \frac{f_\tau(\omega_{l_1} + \omega_{l_2}) - f_\tau(\omega_{l_1})f_\tau(\omega_{l_2})}{\omega_{l_1}} e^{-i(\omega_{l_1} + \omega_{l_2})t} \hat{h}_{l_2} \hat{h}_{l_1}. \quad (\text{C3})$$

In fact, at high orders in the perturbative expansion, cancellations of this kind take place on a very large scale (see App. C2), which makes Eq.(C1) inefficient for computation. In fact, we show in App. C2 that after the mass cancellation takes place, all operator products of the form $\hat{h}_{\mu_1} \cdots \hat{h}_{\mu_1} \rho \hat{h}_{\nu_1} \cdots \hat{h}_{\nu_r}$ have the same harmonic time-dependence $\omega = \sum_i \mu_i + \sum_j \nu_j$.

In what follows, we generalize the two-point contractions defined in Lee et al. [9] to multi-point contractions. According to Eq.(C1), $\mathcal{L}_k\bar{\rho}$ can always be written as the sum of a collection of nested averages of $\hat{H}(t)$, $\hat{U}_i(t)$, and $\hat{U}_j^\dagger(t)$ multiplied to either sides of $\bar{\rho}$, where $\hat{U}_i(t)$ and $\hat{U}_j^\dagger(t)$ can only appear on the left and right side of $\bar{\rho}$ respectively, while $\hat{H}$ is always present and can appear at either end of the product. Therefore, as discussed in the main text, we can group terms in $\mathcal{L}_k$ by the shape of the nested averages so that

$$\mathcal{L}_k\bar{\rho}(t) = \sum_{r=0}^{k-1} \mathcal{W}_{k-r,r}(t)[\bar{\rho}] - h.c. \quad (\text{C4a})$$

for any $k \geq 1$, where the contraction $\mathcal{W}_{l,r}(t)[\bar{\rho}]$ is the sum of all nested averages that have $l(r)$ operators to the left(right) of $\bar{\rho}$ with the first operator being $\hat{H}$.

Plugging Eq.(C4) into the recurrence relation C1 and grouping together all terms that have the weight (l, r) with $l + r = k$, we find the following recurrence relation for these contraction super-operators

$$\mathcal{W}_{0,0} \equiv 0 \quad (\text{C5a})$$

$$\mathcal{W}_{k,0}[\bar{\rho}] = \overline{HU_{k-1}}\bar{\rho} - \sum_{k'=1}^{k-1} \mathcal{W}_{k',0}[\overline{U_{k-k'}}\bar{\rho}] \quad (\text{C5b})$$

$$\mathcal{W}_{l,k-l}(t)[\bar{\rho}] = \overline{HU_{l-1}\bar{\rho}U_{k-l}^\dagger} - \sum_{k'=1}^{k-1} \sum_{l'=\max(0,l-k')}^{\min(l-1,k-k')} \mathcal{W}_{l-l',k'-(l-l')} \left[\overline{U_{l'}\bar{\rho}U_{k-k'-l'}^\dagger} \right] \quad (\text{C5c})$$

where the second equation is just a special case of the third equation, given here for clarity only. The contraction in the second sum of the last two equations are to be understood as operating on the inner time-averaged product, multiplying operators to the left and/or to the right of the time-average. The recurrence relation in Eq.(C5) can be confirmed by plugging it back into Eq.(C1) together with the decomposition in Eq.(C4).

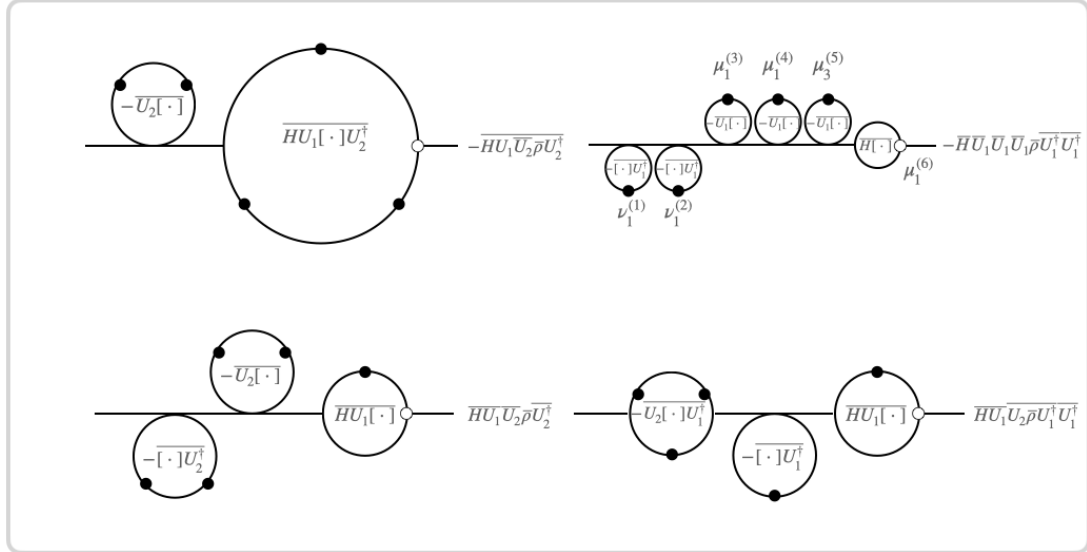


Figure 1. A couple of example bubble diagrams contributing to the contraction super-operator $\mathcal{W}_{4,2}[\bar{\rho}]$. The time-averaged (super)operator inside the bubble represents the factor the bubble gives rise to, and the nested average to the right of the bubble shows the total contribution from that diagram. The diagram is read left to right, with the leftmost bubbles operating closest to $\bar{\rho}$ and the subsequent bubbles operating on the result of previous ones, appearing further away from $\bar{\rho}$.

Looking at the contractions as defined through the recurrence relation in Eq.(C5), we see that they are always of the following form,

$$\mathcal{W}_{l,r}[\bar{\rho}] = \sum_{d \in \text{diagrams}(l,r)} \overline{\hat{H}\hat{U}_{l_{\|d\|}-1}[\cdot]\hat{U}_{r_{\|d\|}}^\dagger} \prod_{n=1}^{\|d\|-1} \left(-\hat{U}_{l_n}[\cdot]\hat{U}_{r_n}^\dagger \right) \bar{\rho} = \overline{\hat{H}\hat{U}_{l-1}\bar{\rho}\hat{U}_r^\dagger} - \overline{\hat{H}\hat{U}_{l-2}\hat{U}_1\bar{\rho}\hat{U}_r^\dagger} + \dots \quad (\text{C6})$$

where “diagrams(l, r)” is a partition-valued function which represents the different ways in which the averages may be arranged by the different natural-number partitions of the 2-tuple (l, r) . These different partitions can be represented visually by “bubble diagram”, where each bubble corresponds to the n -th part of a particular partition d . Namely, for the n -th bubble we would have a 2-tuple (l_n, r_n) such that $\sum_{n=1}^{\|d\|} l_n = l$ and $\sum_{n=1}^{\|d\|} r_n = r$.

Fig. 1, shows a couple of examples of the bubble-diagram visualization, where the dots in a bubble represent the Hamiltonian operators appearing in the corresponding time average. In particular, dots in the upper (lower) half of the diagram represent operators to the left (right) of the density matrix. A contraction with weight (l, r) can then be represented by the set of all bubble diagrams with l upper operators and r lower operators inserted into the bubbles. Finally, the hollow dot represents the leftmost $\hat{H}$ term in the product.

Note that the contractions are not symmetric with respect to the weights, since there is always an $\hat{H}$ term on the left. For example, a $\mathcal{W}_{3,1}[\bar{\rho}]$ contraction super-operator would be a sum over the following time-averaged terms

$$\begin{aligned} \mathcal{W}_{3,1}[\bar{\rho}] \equiv \overline{H\hat{U}_2\bar{\rho}U_1^\dagger} = \\ \overline{HU_2\bar{\rho}U_1^\dagger} - \overline{H\bar{U}_2\bar{\rho}U_1^\dagger} - \overline{H\bar{U}_2\bar{\rho}U_1^\dagger} - \overline{H\bar{U}_2\bar{\rho}U_1^\dagger} + 2\overline{H\bar{U}_2\bar{\rho}U_1^\dagger} \\ - \overline{HU_1\bar{U}_1\bar{\rho}U_1^\dagger} - \overline{H\bar{U}_1U_1\bar{\rho}U_1^\dagger} + 2\overline{H\bar{U}_1\cdot\bar{U}_1\bar{\rho}U_1^\dagger} + \overline{H\bar{U}_1\bar{U}_1\bar{\rho}U_1^\dagger} \\ + \overline{H\bar{U}_1U_1\bar{\rho}U_1^\dagger} + \overline{H\bar{U}_1\bar{U}_1\bar{\rho}U_1^\dagger} - 3\overline{H\bar{U}_1\bar{U}_1\bar{\rho}U_1^\dagger} \end{aligned}$$

where we have generalized the contraction notation in Lee et al. [9] as another representation of $\mathcal{W}_{3,1}[\bar{\rho}]$ in the first equality. These are all the contributions appearing in the TCG master equation with weight $(3, 1)$, where the weight implies the number of Hamiltonian operators in $\hat{H}$ appearing on either side of the density matrix.

2. Proving the harmonic time-dependence of the contractions

To derive the formulas for the corrections, we make the ansatz that the contraction super-operators can be written as a sum of operator product with only a harmonic time-dependence,

$$\mathcal{W}_{l,r}(t)[\bar{\rho}] = \sum_{(\vec{\mu}, \vec{\nu}) \in \mathcal{P}_{l,r}[\Omega]} C_{l,r}(\vec{\mu}, \vec{\nu}) e^{-i(\sum_i^l \mu_i + \sum_i^r \nu_i)t} \cdot \hat{h}_{\mu_1} \cdots \hat{h}_{\mu_l} \hat{\rho} \hat{h}_{\nu_1} \cdots \hat{h}_{\nu_r} \quad (\text{C7})$$

where the contraction coefficients $C_{l,r}(\vec{\mu}, \vec{\nu})$ are time-independent functions of the inserted frequencies, and they effectively encode all the diagram contributions.

Additionally, in Equation C6 we have shown that the contraction super-operators can be written as a sum of products of averages:

$$\mathcal{W}_{l,r}[\bar{\rho}] = \sum_{d \in \text{diagrams}(l,r)} \alpha_d \cdot \overline{HU_{l_{\|d\|}-1}[\cdot]U_{r_{\|d\|}}^\dagger} \prod_{n=1}^{\|d\|-1} \overline{U_{l_n}[\cdot]U_{r_n}^\dagger} \bar{\rho} = \overline{HU_{l-1}\bar{\rho}U_r^\dagger} + \overline{HU_{l-2}\bar{U}_1\bar{\rho}U_r^\dagger} + \cdots$$

More specifically, each nested average in the sum can be obtained by repeatedly acting super-operators of the form $\overline{U_l[\cdot]U_r^\dagger}$ on $\bar{\rho}$ before finally acting the last super-operator $\overline{HU_{l-1}[\cdot]U_r^\dagger}$ onto the preceding result. Using these two assumptions, and by calculating these averages, we can derive an expression for the contraction coefficients.

We start by assuming that the Hamiltonian can be written as a discrete sum of Harmonic terms $\hat{H} = \sum_j \hat{h}_j e^{-i\omega_j t}$, where the coupling strengths have been absorbed into the Hamiltonian operators $\hat{h}_j$. We can derive an analytical expression for $\hat{U}_k$ in the expression of the unitary evolution operator:

$$\hat{U}_n(t) = -i \int_0^t dt' \hat{H}(t') \hat{U}_{n-1}(t') = (-i)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_1 \cdots dt_n \cdot \hat{H}(t_1) \cdots \hat{H}(t_n). \quad (\text{C8})$$

Plugging in the expansion of the Hamiltonian gives a simpler integral and allows us to decouple the limits of integration

$$\hat{U}_n(t) = \sum_{\omega_1, \dots, \omega_n \in \Omega} \hat{h}_{\omega_1} \cdots \hat{h}_{\omega_n} \cdot (-i)^n \int_0^t dt_1 e^{-i\omega_1 t_1} \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdot e^{-i\omega_n t_n}. \quad (\text{C9})$$

The last integral over t_n is fully decoupled, allowing us to calculate it explicitly:

$$\int_0^{t_n-1} dt_n \cdot e^{-i\omega_n t_n} = -\frac{1}{i\omega_n} [e^{-i\omega_n t_{n-1}} - 1] \quad (\text{C10})$$

Plugging it back into Eq.(C9), we see that there is now a sum of two terms that have the same structure of integrals but with different frequencies. If we keep doing that n times, we end up with

$$U_l = \sum_{\vec{\mu}} \sum_{j_l=0}^l \frac{(-1)^{l+j_l} e^{-i \sum_{k=0}^{j_l-1} \mu_{l-k} t}}{\vec{\mu}[l-j_l+1:l]! \vec{\mu}[l-j_l:1]!} h_{\mu_l} \cdots h_{\mu_1} \quad (\text{C11})$$

$$\begin{aligned} U_r^\dagger &= \sum_{\vec{\nu}} \sum_{j_r=0}^{j_r} \frac{(-1)^{r+j_r} e^{-i \sum_{k=0}^{j_r-1} \nu_{r-k} t}}{(-\vec{\nu}[r-j_r+1:r]!) (-\vec{\nu}[r-j_r:1]!)} h_{\nu_1} \cdots h_{\nu_r} \\ &= \sum_{\vec{\nu}} \sum_{j_r=0}^r \frac{(-1)^{j_r} e^{-i \sum_{k=0}^{j_r-1} \nu_{r-k} t}}{\vec{\nu}[r-j_r+1:r]! \vec{\nu}[r-j_r:1]!} h_{\nu_1} \cdots h_{\nu_r} \end{aligned} \quad (\text{C12})$$

and

$$HU_l = \sum_{\vec{\mu}} \sum_{j_l=0}^l \frac{(-1)^{l+j_l} e^{-i \sum_{k=0}^{j_l-1} \mu_{l+1-k} t}}{\vec{\mu}[l-j_l+1:l]! \vec{\mu}[l-j_l:1]!} h_{\mu_{l+1}} h_{\mu_l} \cdots h_{\mu_1} \quad (\text{C13})$$

where $\vec{\mu}[i, j] \equiv (\mu_i, \mu_{i+1}, \dots, \mu_j)$ and we define $\vec{\mu}!$ to be the “vector-factorial” so that

$$\vec{\mu}! \equiv (\mu_1, \mu_2, \dots, \mu_{\|\mu\|})! = (\mu_1 + \mu_2 + \dots + \mu_{\|\mu\|}) \cdots (\mu_1 + \mu_2) \cdot \mu_1, \quad (\text{C14})$$

with $\|\mu\|$ indicating the number of elements in $\vec{\mu}$. So for example, $(\mu_1, \mu_2, \mu_3)! = (\mu_1 + \mu_2 + \mu_3) \cdot (\mu_1 + \mu_2) \cdot \mu_1$. Therefore, we have the following expression for a typical bubble factor (here “typical” means that it is not the special bubble at the right end of the diagram):

$$\begin{aligned} -\overline{U_l[\cdot] U_r^\dagger} &= - \sum_{\vec{\mu}, \vec{\nu}} \sum_{j_l=0}^l \sum_{j_r=0}^r \frac{(-1)^{l+j_l+j_r} f_\tau(\sum_{k=0}^{j_l-1} \mu_{l-k} + \sum_{k=0}^{j_r-1} \nu_{r-k}) e^{-i(\sum_{k=0}^{j_l-1} \mu_{l-k} + \sum_{k=0}^{j_r-1} \nu_{r-k})t}}{\vec{\mu}[l-j_l+1:l]! \vec{\mu}[l-j_l:1]! \vec{\nu}[r-j_r+1:r]! \vec{\nu}[r-j_r:1]!} \\ &\quad \cdot h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r} \\ &= \sum_{\vec{\mu}, \vec{\nu}} \sum_{j_l=0}^l \sum_{j_r=0}^r \left(-B^{j_l, j_r}(\vec{\mu}, \vec{\nu}) \right) h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r} \end{aligned} \quad (\text{C15})$$

where,

$$B^{j_l, j_r}(\vec{\mu}, \vec{\nu}) \equiv \frac{(-1)^{l+j_l+j_r} f_\tau(\sum_{k=0}^{j_l-1} \mu_{l-k} + \sum_{k=0}^{j_r-1} \nu_{r-k}) e^{-i(\sum_{k=0}^{j_l-1} \mu_{l-k} + \sum_{k=0}^{j_r-1} \nu_{r-k})t}}{\vec{\mu}[l-j_l+1:l]! \vec{\mu}[l-j_l:1]! \vec{\nu}[r-j_r+1:r]! \vec{\nu}[r-j_r:1]!} \quad (\text{C16})$$

with l and r being the lengths of $\vec{\mu}$ and $\vec{\nu}$ respectively; similarly, the special bubble at the right end gives rise to the factor,

$$\begin{aligned} \overline{HU_l[\cdot] U_r^\dagger} &= \sum_{\vec{\mu}, \vec{\nu}} \sum_{j_l=0}^l \sum_{j_r=0}^r \frac{(-1)^{l+j_l+j_r} f_\tau(\sum_{k=0}^{j_l-1} \mu_{l+1-k} + \sum_{k=0}^{j_r-1} \nu_{r-k}) e^{-i(\sum_{k=0}^{j_l-1} \mu_{l+1-k} + \sum_{k=0}^{j_r-1} \nu_{r-k})t}}{\vec{\mu}[l-j_l+1:l]! \vec{\mu}[l-j_l:1]! \vec{\nu}[r-j_r+1:r]! \vec{\nu}[r-j_r:1]!} \\ &\quad \cdot h_{\mu_{l+1}} h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r} \\ &\equiv \sum_{\vec{\mu}, \vec{\nu}} \sum_{j_l=0}^l \sum_{j_r=0}^r \tilde{B}^{j_l, j_r}(\vec{\mu}, \vec{\nu}) \cdot h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r}. \end{aligned} \quad (\text{C17})$$

where we define,

$$\tilde{B}^{j_l, j_r}(\vec{\mu}, \vec{\nu}) \equiv \frac{(-1)^{l+j_l+j_r} f_\tau(\sum_{k=0}^{j_l} \mu_{l+1-k} + \sum_{k=0}^{j_r-1} \nu_{r-k}) e^{-i(\sum_{k=0}^{j_l} \mu_{l+1-k} + \sum_{k=0}^{j_r-1} \nu_{r-k})t}}{\vec{\mu}[l-j_l+1:l]! \vec{\mu}[l-j_l:1]! \vec{\nu}[r-j_r+1:r]! \vec{\nu}[r-j_r:1]!}. \quad (\text{C18})$$

Therefore, we see that for each combination of inserted operators, the contributing factor of a particular bubble can be written as a sum of terms each of which oscillates at the total frequency of a subset of neighboring frequencies inserted on the right side of the bubble, as indicated by the blue dots in Fig. 2.

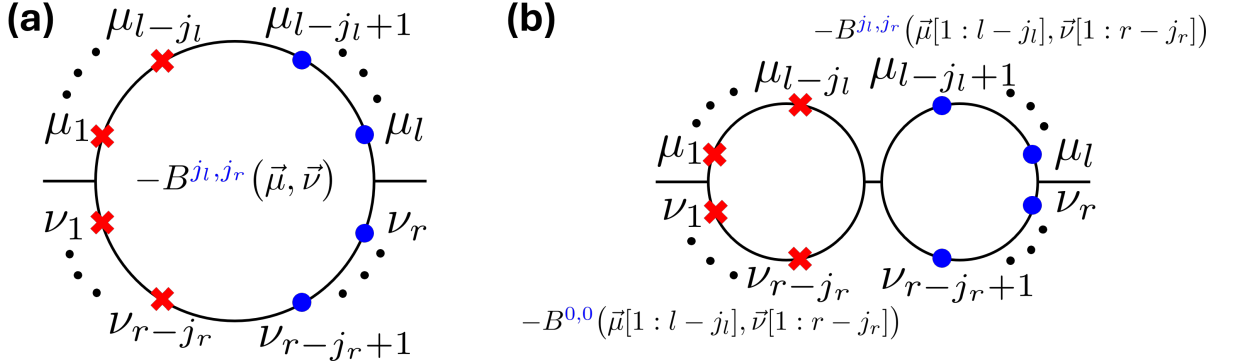


Figure 2. These color-coded bubbles correspond to individual contributing terms oscillating at the total frequency of the blue dots on the right side of the bubbles. As shown in Eq.(C15) and Eq.(C19), one has the identity $B^{j_l, j_r}(\vec{\mu}, \vec{\nu}) = B^{0,0}(\vec{\mu}[1:l-j_l], \vec{\nu}[1:r-j_r]) B^{j_l, j_r}(\vec{\mu}[1:l-j_l], \vec{\nu}[1:r-j_r])$, which implies that the bubble contributions in panel (a) and panel (b) cancel each other out exactly.

On the other hand, we also have the following formula for the contributions of two adjacent typical bubbles:

$$\begin{aligned} & \left(-\overline{U_{j_l}[\cdot]} U_{j_r}^\dagger \right) \left(-\overline{U_{l-j_l}[\cdot]} U_{r-j_r}^\dagger \right) \\ &= \sum_{\vec{\mu}, \vec{\nu}} \frac{(-1)^{l-j_r}}{\vec{\mu}[l-j_l:1]! \vec{\nu}[r-j_r:1]!} \cdot \frac{(-1)^{j_l} f_\tau(\sum_{k=0}^{j_l-1} \mu_{l-k} + \sum_{k=0}^{j_r-1} \nu_{r-k}) e^{-i(\sum_{k=0}^{j_l-1} \mu_{l-k} + \sum_{k=0}^{j_r-1} \nu_{r-k})t}}{\vec{\mu}[l-j_l+1:l]! \vec{\nu}[r-j_r+1:r]!} \\ &+ \sum_{\vec{\mu}, \vec{\nu}} \sum_{(n_l, n_r, m_l, m_r) \neq (0,0,j_l,j_r)}^{(l-j_l, r-j_r, j_l, j_r)} \frac{(-1)^{l-j_r+n_l+n_r} f_\tau(\sum_{k=0}^{n_l-1} \mu_{l-j_l-k} + \sum_{k=0}^{n_r-1} \nu_{r-j_r-k})}{\vec{\mu}[l-j_l-n_l+1:l-j_l]! \vec{\mu}[l-j_l-n_l:1]!} \\ &\quad \cdot \frac{e^{-i(\sum_{k=0}^{n_l-1} \mu_{l-j_l-k} + \sum_{k=0}^{n_r-1} \nu_{r-j_r-k})t}}{\vec{\nu}[r-j_r-n_r+1:r-j_r]! \vec{\nu}[r-j_r-n_r:1]!} \\ &\quad \cdot \frac{(-1)^{j_r+m_l+m_r} f_\tau(\sum_{k=0}^{m_l-1} \mu_{l-k} + \sum_{k=0}^{m_r-1} \nu_{r-k}) e^{-i(\sum_{k=0}^{m_l-1} \mu_{l-k} + \sum_{k=0}^{m_r-1} \nu_{r-k})t}}{\vec{\mu}[l-m_l+1:l]! \vec{\mu}[l-m_l:l-j_l+1]! \vec{\nu}[r-m_r+1:r]! \vec{\nu}[r-m_r:r-j_r+1]!} \\ &\quad \cdot h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r} \\ &= \sum_{\vec{\mu}, \vec{\nu}} B^{j_l, j_r}(\vec{\mu}, \vec{\nu}) \cdot h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r} \\ &+ \sum_{\vec{\mu}, \vec{\nu}} \sum_{(n_l, n_r, m_l, m_r) \neq (0,0,j_l,j_r)} B^{n_l, n_r}(\vec{\mu}[1:l-j_l], \vec{\nu}[1:r-j_r]) \cdot B^{m_l, m_r}(\vec{\mu}[l-j_l+1:l], \vec{\nu}[r-j_r+1:r]) \\ &\quad \cdot h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r} \end{aligned} \quad (\text{C19})$$

and similarly, for a typical bubble adjacent to the special bubble at the right end we have

$$\begin{aligned}
& \left(-HU_{j_l}[\cdot]U_{j_r}^\dagger \right) \left(-U_{l-j_l}[\cdot]U_{r-j_r}^\dagger \right) \\
&= \sum_{\vec{\mu}, \vec{\nu}} \frac{(-1)^{l-j_r}}{\vec{\mu}[l-j_l:1]!\vec{\nu}[r-j_r:1]!} \cdot \frac{(-1)^{j_l} f_\tau(\sum_{k=0}^{j_l-1} \mu_{l-k} + \sum_{k=0}^{j_r-1} \nu_{r-k}) e^{-i(\sum_{k=0}^{j_l-1} \mu_{l-k} + \sum_{k=0}^{j_r-1} \nu_{r-k})t}}{\vec{\mu}[l-j_l+1:l]!\vec{\nu}[r-j_r+1:r]!} \\
&+ \sum_{\vec{\mu}, \vec{\nu}} \sum_{(n_l, n_r, m_l, m_r) \neq (0,0,j_l,j_r)}^{(l-j_l, r-j_r, j_l, j_r)} \frac{(-1)^{l-j_r+n_l+n_r} f_\tau(\sum_{k=0}^{n_l-1} \mu_{l-j_l-k} + \sum_{k=0}^{n_r-1} \nu_{r-j_r-k})}{\vec{\mu}[l-j_l-n_l+1:l-j_l]!\vec{\mu}[l-j_l-n_l:1]!} \\
&\quad \cdot \frac{e^{-i(\sum_{k=0}^{n_l-1} \mu_{l-j_l-k} + \sum_{k=0}^{n_r-1} \nu_{r-j_r-k})t}}{\vec{\nu}[r-j_r-n_r+1:r-j_r]!\vec{\nu}[r-j_r-n_r:1]!} \\
&\quad \cdot \frac{(-1)^{j_r+m_l+m_r} f_\tau(\sum_{k=0}^{m_l-1} \mu_{l-k} + \sum_{k=0}^{m_r-1} \nu_{r-k}) e^{-i(\sum_{k=0}^{m_l-1} \mu_{l-k} + \sum_{k=0}^{m_r-1} \nu_{r-k})t}}{\vec{\mu}[l-m_l+1:l]!\vec{\mu}[l-m_l:l-j_l+1]!\vec{\nu}[r-m_r+1:r]!\vec{\nu}[r-m_r:r-j_r+1]!} \\
&\quad \cdot h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r} \\
&= \sum_{\vec{\mu}, \vec{\nu}} \tilde{B}^{j_l, j_r}(\vec{\mu}, \vec{\nu}) \cdot h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r} \\
&+ \sum_{\vec{\mu}, \vec{\nu}} \sum_{(n_l, n_r, m_l, m_r) \neq (0,0,j_l,j_r)} B^{n_l, n_r}(\vec{\mu}[1:l-j_l], \vec{\nu}[1:r-j_r]) \\
&\quad \cdot \tilde{B}^{m_l, m_r}(\vec{\mu}[l-j_l+1:l+1], \vec{\nu}[r-j_r+1:r]) \cdot h_{\mu_l} \cdots h_{\mu_1}[\cdot] h_{\nu_1} \cdots h_{\nu_r}
\end{aligned} \tag{C20}$$

In fact, according to Fig. 2, only two kinds of typical bubble contributions cannot be canceled in this fashion: the contributing term is either a constant in time (only red crosses are inserted) or oscillates at the total bubble frequency (only blue dots are inserted), which we refer to as “empty” or “full” bubbles respectively. Following the same procedure, one can also show that the contributions from the special bubble at the right end can also be canceled by bubble-splitting unless it is either full or empty (here the contribution corresponding to the “empty” special bubble oscillates at the frequency μ_l of the special operator at the right end, instead of being a constant in time).

On the flip side of cancellation by bubble splitting, we see that an empty bubble right to the left of a full bubble can be canceled by merging the two bubbles together. Therefore, the only bubble diagrams surviving this cancellation process are those with full bubbles on the left and empty bubble on the right. However, this implies that there are no empty bubbles, since the special operator inserted on the right end of a diagram is always filled (represented by a blue dot) by construction, as shown in Fig. 3.

Consequently, according to Eq.(C6), we have

$$\begin{aligned}
\mathcal{W}_{l,r}[\bar{\rho}] &= \sum_{d \in \text{diagrams}(l,r)} \left(\tilde{B}^{l_{\parallel d}, r_{\parallel d}}(\vec{\mu}^{(b_{\parallel d})}, \vec{\nu}^{(b_{\parallel d})}) \right) \prod_{n=1}^{\|d\|-1} \left(-B^{l_n, r_n}(\vec{\mu}^{(b_n)}, \vec{\nu}^{(b_n)}) \right) \cdot \hat{h}_{\mu_l} \cdots \hat{h}_{\mu_1} \bar{\rho} \hat{h}_{\nu_1} \cdots \hat{h}_{\nu_r} \\
&= \sum_{d \in \text{diagrams}(l,r)} \left(\sum_i \mu_i^{(b_{\parallel d})} \right) \frac{(-1)^{r_{\parallel d}} f_\tau(\sum_i \mu_i^{(b_{\parallel d})} + \sum_i \nu_i^{(b_{\parallel d})}) e^{-i(\sum_i \mu_i^{(b_{\parallel d})} + \sum_i \nu_i^{(b_{\parallel d})})t}}{\vec{\mu}^{(b_{\parallel d})}! \vec{\nu}^{(b_{\parallel d})}!} \\
&\quad \cdot (-1)^{\|d\|-1} \prod_{n=1}^{\|d\|-1} \frac{(-1)^{r_n} f_\tau(\sum_i \mu_i^{(b_n)} + \sum_i \nu_i^{(b_n)}) e^{-i(\sum_i \mu_i^{(b_n)} + \sum_i \nu_i^{(b_n)})t}}{\vec{\mu}^{(b_n)}! \vec{\nu}^{(b_n)}!} \\
&\quad \cdot \hat{h}_{\mu_l} \cdots \hat{h}_{\mu_1} \bar{\rho} \hat{h}_{\nu_1} \cdots \hat{h}_{\nu_r} \\
&= \left[\sum_{d \in \text{diagrams}(l,r)} (-1)^{r+\|d\|-1} \left(\sum_i \mu_i^{(b_{\parallel d})} \right) \prod_{b \in d} \frac{f_\tau(\sum_i \mu_i^{(b)} + \sum_i \nu_i^{(b)})}{\vec{\mu}^{(b)}! \vec{\nu}^{(b)}!} \right] e^{-i(\sum_{i=1}^l \mu_i + \sum_{i=1}^r \nu_i)t} \\
&\quad \cdot \hat{h}_{\mu_l} \cdots \hat{h}_{\mu_1} \bar{\rho} \hat{h}_{\nu_1} \cdots \hat{h}_{\nu_r}
\end{aligned} \tag{C21}$$

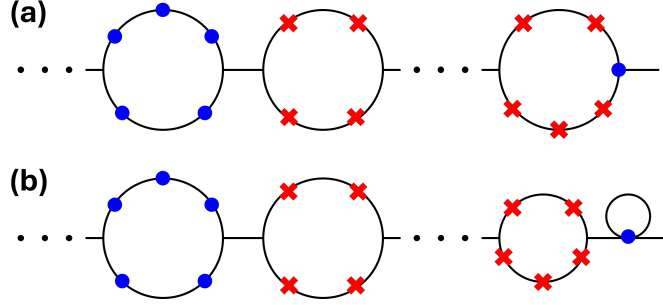


Figure 3. For a diagram not to be canceled by splitting partially filled bubbles or by merging adjacent empty and full bubbles, it has to contain no partially filled bubbles, and all its empty bubbles have to be on the right side of the diagram. However, by our construction, the special operator on the right end of a diagram comes from the un-integrated Hamiltonian H and is therefore always filled (i.e., it always contributes the time-dependent factor $e^{-i\mu_l t}$ corresponding to its frequency μ_l). Therefore, the diagrams shown in panel (a) and panel (b) cancel each other out.

Or equivalently,

$$\mathcal{W}_{l,r}(t)[\bar{\rho}] = \sum_{(\vec{\mu}, \vec{\nu}) \in \mathcal{P}_{l,r}[\Omega]} C_{l,r}(\vec{\mu}, \vec{\nu}) e^{-i(\sum_{i=1}^l \mu_i + \sum_{i=1}^r \nu_i)t} \cdot \hat{h}_{\mu_l} \cdots \hat{h}_{\mu_1} \bar{\rho} \hat{h}_{\nu_1} \cdots \hat{h}_{\nu_r}.$$

with

$$C_{l,r}(\vec{\mu}, \vec{\nu}) = \sum_{d \in \text{diagrams}(l,r)} (-1)^{r+\|d\|-1} \left(\sum_i \mu_i^{(b_{\|d\|})} \right) \prod_{b \in d} \frac{f_\tau \left(\sum_i \mu_i^{(b)} + \sum_i \nu_i^{(b)} \right)}{\vec{\mu}^{(b)}! \vec{\nu}^{(b)}!},$$

which proves the formulas for the contraction coefficients the main text. Noticeably, the mass cancellation of the different frequency terms is due to the summation of all diagram contributions, leaving only a single harmonic contribution for each diagram; in particular, the special operator due to H inserted on the right end of each diagram is crucial for this mass cancellation to take place.

3. Symmetries and separation into Hamiltonian and pseudo-dissipators

We notice that Eq.(C 2) reveals some important symmetries of the contraction coefficients. These symmetries keep the last element of $\vec{\mu}$ constant as we permute the vectors,

$$C_{l,r} = C_{l,r}((\mu_1, \cdots, \mu_{l-1}, \mu_l), (\nu_1, \cdots, \nu_r)) \quad (\text{C22})$$

To respect the symmetries of $C_{l,r}$, permutations of the frequencies must keep ω_1 as the last element of $\vec{\mu}$. For example, with $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_l)$ and $\vec{\nu} = (\nu_1, \nu_2, \cdots, \nu_r)$, one has

$$C_{l,r}(-\vec{\mu}, -\vec{\nu}) = (-1)^{l+r-1} C_{l,r}(\vec{\mu}, \vec{\nu}) \quad (\text{C23a})$$

$$C_{l,r}(\vec{\mu}, \vec{\nu}) = C_{r+1, l-1}(-(\nu_1, \nu_2, \cdots, \nu_r, \mu_l), -(\mu_1, \mu_2, \cdots, \mu_{l-1})). \quad (\text{C23b})$$

where the symmetry in the second identity reflects swapping the upper modes and lower modes, while keeping the special Hamiltonian mode contribution μ_l unaffected.

$$C_{l,r}(\vec{\mu}, \vec{\nu}) = C_{r+1, l-1}(-(\vec{\nu} \oplus_{r+1} \mu_l), -(\vec{\mu} \ominus_l \mu_l)). \quad (\text{C24})$$

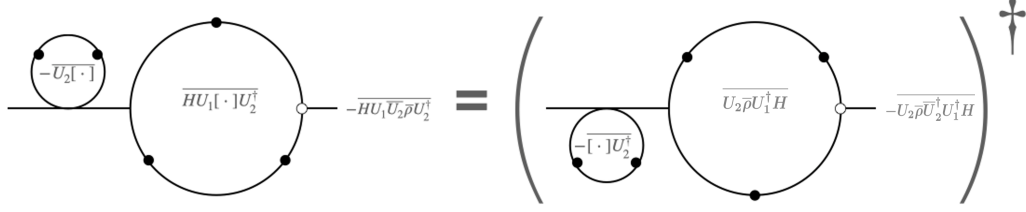


Figure 4. The symmetries of the contraction coefficients reflect the hermicity as well as the trace-preserving property of the underlying time evolution. In terms of the diagrams, they manifest as a symmetry for reflection about the main axis of the diagram, and the fact that the special frequency μ_l from the H operator does not participate in the vector factorials in Eq.(C 2) and may therefore be moved from the right of $\bar{\rho}$ to the left of $\bar{\rho}$ if the overall sign is properly adjusted. However, the time reversal symmetry can be spontaneously broken if some of the inserted operators assume nonzero expectation values or if the regularization of apparently singular bubbles produces imaginary prefactors.

where $\oplus_{r+1}$ denotes concatenation at index $r+1$ and $\ominus_l$ denotes removing from position l , producing vectors with length $r+1$ and $l-1$ respectively.

This symmetry reflects the mirror symmetry of the diagrams around the main axis, as shown in Fig. 4, which in turn is a manifestation of the hermicity of the terms. In particular, the second identity is not only useful for deriving the analytical expression of $C_{l,r}$ from that of $C_{r+1,l-1}$, but also necessary for the partial Liouvillian $\mathcal{L}_k$ to be trace-preserving at all orders in the perturbation theory, by virtue of the Lindblad-like form of $\mathcal{L}_k$. Let us write the TCG Liouvillian again,

$$\mathcal{L}_k(t)\bar{\rho} = \sum_{l=1}^k \mathcal{W}_{l,k-l}[\bar{\rho}] - h.c. \quad (C25)$$

$$= \sum_{l=1}^k \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{l,k-l}[\Omega]} \left(C_{l,r}(\vec{\mu}, \vec{\nu}) s_{\vec{\mu}^{\text{rev}}} \bar{\rho} s_{\vec{\nu}} - C_{l,r}(\vec{\mu}, \vec{\nu}) s_{\vec{\nu}}^\dagger \bar{\rho} s_{\vec{\mu}^{\text{rev}}}^\dagger \right) \quad (C26)$$

where we have absorbed the harmonic time-dependent factors as well as the coupling strengths into the operator $s_{\vec{\nu}}$ for brevity:

$$s_{\vec{\nu}} = \left(\prod_{\omega \in \vec{\nu}} e^{-i\omega t} \right) h_{\nu_1} h_{\nu_2} \cdots h_{\nu_r}. \quad (C27)$$

Since we sum over all frequencies in the set Ω and all vector lengths

$$(l, r) \in \{(1, k-1), (2, k-2), \dots, (k, 0)\},$$

we can make the following dummy variable substitution in the second term of the right-hand side of Eq.(C26) for all $1 \leq l \leq k-1$:

$$\vec{\mu} \rightarrow -\vec{\nu} \quad (C28)$$

$$\vec{\nu} \rightarrow -\vec{\mu} \quad (C29)$$

whereas for $l = k$ we make the substitution $\vec{\mu} \rightarrow -\vec{\mu}^{\text{rev}}$. This allows us to write the sum in terms of a single operator,

$$\begin{aligned} \mathcal{L}_k(t)\bar{\rho} = & \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} C_{k,0}(\vec{\mu}, \emptyset) s_{\vec{\mu}^{\text{rev}}} \bar{\rho} - \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} C_{k,0}(-\vec{\mu}^{\text{rev}}, \emptyset) \bar{\rho} s_{\vec{\mu}^{\text{rev}}} \\ & + \sum_{l=1}^{k-1} \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{l,k-l}[\Omega]} \left(C_{l,r}(\vec{\mu}, \vec{\nu}) - C_{l,r}(-\vec{\nu}, -\vec{\mu}) \right) s_{\vec{\mu}^{\text{rev}}} \bar{\rho} s_{\vec{\nu}} \end{aligned} \quad (C30)$$

where we have used the identity $s_{-\vec{\mu}}^\dagger = s_{\vec{\mu}}$, and then divided the sum into $l = k$ and $l \neq k$ terms. Now we note that we can rearrange the terms into dissipators and commutators, with the help of the following identities:

$$s_{\vec{\mu}}^{\text{rev}} \bar{\rho} s_{\vec{\nu}} = \mathcal{D}[s_{\vec{\mu}}^{\text{rev}}, s_{\vec{\nu}}] \bar{\rho} + \frac{1}{2} \{s_{\vec{\nu}} s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} \quad (\text{C31})$$

$$s_{\vec{\mu}}^{\text{rev}} \bar{\rho} = \frac{1}{2} ([s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}] + \{s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\}) \quad (\text{C32})$$

$$\bar{\rho} s_{\vec{\mu}}^{\text{rev}} = \frac{1}{2} (-[s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}] + \{s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\}). \quad (\text{C33})$$

Plugging the identities above into Eq.(C30), we have

$$\begin{aligned} \mathcal{L}_k(t) \bar{\rho} = & \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} \frac{C_{k,0}(\vec{\mu}, \emptyset) + C_{k,0}(-\vec{\mu}^{\text{rev}}, \emptyset)}{2} [s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}] \\ & + \sum_{l=1}^{k-1} \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{l,k-l}[\Omega]} \left(C_{l,r}(\vec{\mu}, \vec{\nu}) - C_{r,l}(-\vec{\nu}, -\vec{\mu}) \right) \mathcal{D}[s_{\vec{\mu}}^{\text{rev}}, s_{\vec{\nu}}] \bar{\rho} \\ & + \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} \frac{C_{k,0}(\vec{\mu}, \emptyset) - C_{k,0}(-\vec{\mu}^{\text{rev}}, \emptyset)}{2} \{s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} \\ & + \sum_{l=1}^{k-1} \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{l,k-l}[\Omega]} \frac{C_{l,r}(\vec{\mu}, \vec{\nu}) - C_{r,l}(-\vec{\nu}, -\vec{\mu})}{2} \{s_{\vec{\nu}} s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} \end{aligned} \quad (\text{C34})$$

where by the symmetry relation in Eq.(C24), we can rewrite the anti-commutator terms in the last two lines of Eq.(C34) as

$$\begin{aligned} & \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} \frac{C_{k,0}(\vec{\mu}, \emptyset) - C_{k,0}(-\vec{\mu}^{\text{rev}}, \emptyset)}{2} \{s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} \\ & + \sum_{l=1}^{k-1} \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{l,k-l}[\Omega]} \frac{C_{l,r}(\vec{\mu}, \vec{\nu}) - C_{l+1,r-1}(\vec{\mu} \oplus_l \nu_r, \vec{\nu} \ominus_r \nu_r)}{2} \{s_{\vec{\nu}} s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} \\ = & - \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} \frac{C_{k,0}(-\vec{\mu}^{\text{rev}}, \emptyset)}{2} \{s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} + \sum_{l=1}^k \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{l,k-l}[\Omega]} \frac{C_{l,r}(\vec{\mu}, \vec{\nu})}{2} \{s_{\vec{\nu}} s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} \\ & - \sum_{l=2}^k \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{l,k-l}[\Omega]} \frac{C_{l,r}(\vec{\mu}, \vec{\nu})}{2} \{s_{\vec{\nu}} s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} \\ = & - \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} \frac{C_{k,0}(-\vec{\mu}^{\text{rev}}, \emptyset)}{2} \{s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} + \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{1,k-1}[\Omega]} \frac{C_{1,k-1}(\vec{\mu}, \vec{\nu})}{2} \{s_{\vec{\nu}} s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} \\ = & - \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} \frac{C_{k,0}(-\vec{\mu}^{\text{rev}}, \emptyset)}{2} \{s_{\vec{\mu}}^{\text{rev}}, \bar{\rho}\} + \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{1,k-1}[\Omega]} \frac{C_{k,0}(-\vec{\nu} \oplus_{k-1} \mu_1, \emptyset)}{2} \{s_{\vec{\nu} \oplus_{k-1} \mu_1}, \bar{\rho}\} = 0 \end{aligned} \quad (\text{C35})$$

where one can make the dummy variable substitution $\vec{\nu} \oplus_{k-1} \mu_1 \rightarrow \vec{\mu}$ in the second term of the last line in order to

make the cancellation manifest. Now that the anti-commutator terms vanish, we are left with

$$\mathcal{L}_k(t)\bar{\rho} = \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{k,0}[\Omega]} \frac{C_{k,0}(\vec{\mu}, \emptyset) + C_{k,0}(-\vec{\mu}^{\text{rev}}, \emptyset)}{2} [\hat{s}_{\vec{\mu}^{\text{rev}}}, \bar{\rho}] \quad (\text{C36})$$

$$+ \sum_{l=1}^{k-1} \sum_{\vec{\mu}, \vec{\nu} \in \mathcal{P}_{l,k-l}[\Omega]} (C_{l,r}(\vec{\mu}, \vec{\nu}) - C_{r,l}(-\vec{\nu}, -\vec{\mu})) \mathcal{D}[\hat{s}_{\vec{\mu}^{\text{rev}}}, \hat{s}_{\vec{\nu}}] \bar{\rho} \quad (\text{C37})$$

which is separated into dissipator and Hamiltonian terms, proving the results in the main text.

Appendix D: Algorithms and implementation

1. Enumeration of the diagrams

To calculate a contraction coefficient $C_{l,r}$, we first need to identify all the diagrams associated with such a contraction, and then calculate the TCG corrections associated with each diagram. For example, the contraction coefficient $C_{l,r}$ receives contributions from all diagrams that have l left modes and r right modes.

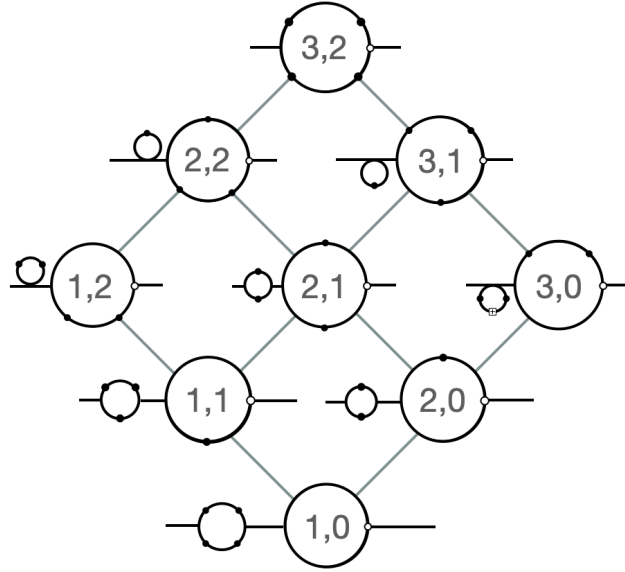


Figure 5. A single level (the first one in this case) in a decomposition tree. It covers all the different ways in which a single-bubble diagram of the form $(3, 2)$ can be decomposed into two-bubble diagrams. The process is repeated on each non-trivial bubble until none is left.

```

1: procedure DIAGRAM_NODE(root_bubble, new_bubble)
2:   if root_bubble.left  $\geq$  1, root_bubble.right  $\geq$  0 then
3:     left_node  $\leftarrow$  diagram_node(root_bubble, (new_bubble.left + 1, new_bubble.right))
4:     right_node  $\leftarrow$  diagram_node(root_bubble, (new_bubble.left, new_bubble.right + 1))
5:     return diagram_node(root_bubble, new_bubble, left_node, right_node)
6:   end if
7: end procedure
8: procedure DECOMPOSE_DIAGRAM(node, decomp_list)
9:   if node can be broken and is not in decomp_list then

```

```

10:      decomp_list ← node.left, node.right
11:      decompose_diagram(node.left, decomp_list)
12:      decompose_diagram(node.right, decomp_list)
13:  end if
14:  return decomp_list
15: end procedure

```

Eq.(C5) establishes a recurrence relation between a contraction coefficient of weight (l, r) and those of lower weights, as elaborated upon in appendix C. This recursive structure bears resemblance to a binary tree, which we refer to as the decomposition tree of the coefficient, as depicted in Fig. 5. For the purpose of systematically examining all possible diagrams, we arrange them in a tree-like structure, with a single-bubble diagram at the pinnacle. A rightward (leftward) step down the tree is equivalent to rupturing the rightmost bubble in the diagram, removing a right-mode (left-mode) from it and incorporating it into a bubble to its left, as illustrated in Fig. 5. This yields all the distinct ways in which we can break a bubble into two bubbles while preserving the same number of left-modes and right-modes. By iterating this process over the newly added bubble, generating another tree, we obtain all the bubble diagrams. The algorithm for this process is outlined in Algorithm 0 and illustrated in Fig. 5.

One caveat is worth noting regarding the handling of the $\hat{H}$ term in the contraction defined by Eq.(C5), which we refer to as the “special mode” and represent by a hollow dot in the diagrams. This term has to be treated in a different manner since the time averages in Eq.(C1) are asymmetrical around the density matrix – the only $\hat{H}$ term always appears as the left-most operator in the contraction. Therefore, the mode associated with the $\hat{H}$ term must remain within the right-most bubble in order to preserve this structure, and is treated differently from the other modes in the calculation of their contributions to the diagram.

2. Calculating singular diagram corrections

Once we have all the different diagrams, calculating the contraction coefficient is a simple matter of summing up the corrections due to each diagram, according to the formula in Eq.(C2). Although the formula is simple enough, it proves problematic to directly implement numerically and symbolically, since the different bubble corrections can diverge when any factor in the vector factorial vanishes. In more exact terms, we can define a set that enumerates these singular points for a bubble i in a diagram,

$$\mathbf{s}_i^{(\mu)} = \{s | \mu_i^1 + \mu_i^2 + \dots + \mu_i^s = 0\} \quad (\text{D1a})$$

$$\mathbf{s}_i^{(\nu)} = \{s' | \nu_i^1 + \nu_i^2 + \dots + \nu_i^{s'} = 0\} \quad (\text{D1b})$$

Defined for a particular bubble correction with a vector of left-modes $\vec{\mu}_i$ and a vector of right-modes $\vec{\nu}_i$. For example, for a bubble left-modes $\mu_i = (0, 1, 2, -3)$ we would have $s_i = \{1, 4\}$, since the first factor is 0 and vanishes trivially, and the fourth factor $0 + 1 + 2 - 3 = 0$ sums up to zero and vanishes as well.

Let us define the contribution due to a single diagram as ϵ_d . In order to obtain the finite contribution near these singular terms, we expand the diagram near the poles by taking each frequency $\omega \rightarrow \omega + d\omega$ and expanding to the order of the pole.

$$\epsilon_d = \epsilon_d^{(0)} \sum_{n_1 \geq 0, u_1 \geq 0, l_1 \geq 0} \dots \sum_{n_{||d||} \geq 0, u_{||d||} \geq 0, l_{||d||} \geq 0} \left(a_{n_i}(\vec{\mu}_i, \vec{\nu}_i) + \frac{b_{u_i}(\vec{\mu}_i) \cdot c_{l_i}(\vec{\nu}_i)}{\prod_{k \in s_i^{(\mu)}, l \in s_i^{(\nu)}} k \cdot l} \right) \times \delta \left(\sum_i (n_i + u_i + l_i) - \sum_i (|s_i| + |s'_i|) = 0 \right) \quad (\text{D2})$$

where the Kronecker-delta δ denotes that the sum is only over solutions to the partition problem in the argument and the sum is over $3||d||$ indices, for example for n_i we have $(n_1, n_2, \dots, n_{||d||})$.

$\epsilon_d^{(0)}$ is just the simple product of the correction factors excluding the resonant terms in the factorial.

$$\epsilon_d^{(0)} = (-1)^{r+||d||-1} \prod_{i=1}^{||d||} \frac{1}{\vec{\mu}_i!^* \cdot \vec{\nu}_i!^*} f\left(\sum_j \left(\mu_i^{(j)} + \nu_i^{(j)}\right)\right) \quad (\text{D3})$$

and we define the non-singular vector factorial as $\vec{v}_i!^* = \prod_{j \notin s_i^{(v)}} (v^1 + \dots + v^j)$.

Additionally, we define the auxiliary functions,

$$a_{n_i}(\vec{\mu}_i, \vec{\nu}_i) = \sum_{k=0}^{\lfloor \frac{n_i}{2} \rfloor} \frac{c(n_i, k)}{n_i!} \tau^{2(n_i-k)} \left(\sum_i \mu_i + \sum_i \nu_i \right)^{n_i-2k} (||\mu_i||_1 + ||\nu_i||_1)^{n_i} \quad (\text{D4a})$$

$$b_{u_i}(\vec{\mu}_i) = \sum_m \prod_{j \notin s_i^{(\mu)}} \left(\frac{-j}{\mu_i^1 + \dots + \mu_i^j} \right)^{m_j} \delta \left(\sum_{j \notin s_i} m_j - u_i \right) \quad (\text{D4b})$$

$$c_{l_i}(\vec{\nu}_i) = \sum_{m_j} \prod_{j \notin s_i} \left(\frac{-j}{\nu_i^1 + \dots + \nu_i^j} \right)^{m_j} \delta \left(\sum_{j \notin s_i^{(\nu)}} m_j - l_i \right) \quad (\text{D4c})$$

$$(\text{D4d})$$

where $c(n, k)$ are the expansion coefficient, defined recursively through,

$$c(0, 0) = 1 \quad (\text{D5a})$$

$$c(n, -1) \equiv 0 \text{ for any } n \quad (\text{D5b})$$

$$c(n, k) \equiv 0 \text{ for } n < 2k \quad (\text{D5c})$$

In that form, the diagram corrections are always finite and well defined, and can be explicitly calculated without relying on explicit limits, which are difficult to calculate numerically.

Appendix E: The TCG effective model – examples

1. Rabi-model contraction coefficients and the effective master equation

These coefficients are determined by a set of diagrams. In this simple example, we have five diagrams contributing to the process, as illustrated in Fig. 6. For the first-order contractions we have a single diagram with only a single mode.

The diagram contributions yield the following coefficients,

$$C_{1,0}(\omega) = e^{-\frac{\omega^2 \tau^2}{2}}. \quad (\text{E1})$$

Similarly, for the second-order contractions we have

$$C_{2,0}(\omega, \omega') = -C_{1,1}(\omega, \omega') = \frac{e^{-\frac{(\omega+\omega')^2 \tau^2}{2}}}{\omega'} - \frac{e^{-\frac{\omega^2 + \omega'^2}{2} \tau^2}}{\omega'}. \quad (\text{E2})$$

Adding up contributions from all the diagrams, we end up with the second-order effective Hamiltonian

$$\hat{H}_{\text{TCG}}^{(2)} = \hat{H}_{\text{r}}^{(2)} + \hat{H}_{\text{cr}}^{(2)} \quad (\text{E3})$$

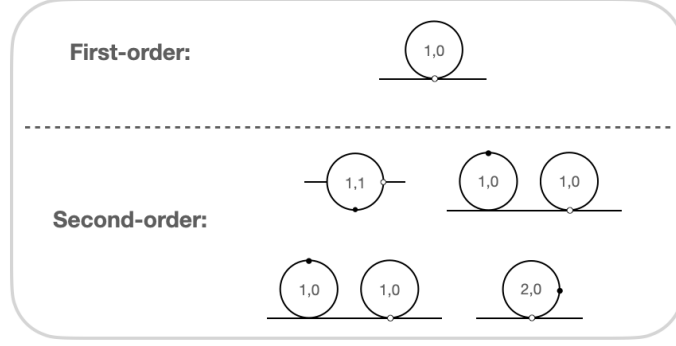


Figure 6. The different diagrams contributing to the TCG super-operators. The top diagram shows the first-order (single-mode) contributions, whereas the bottom diagrams show the second-order (two-mode) contributions.

where the “co-rotating” terms $\hat{H}_r^{(2)}$ are not exponentially suppressed in the $\omega_a \sim \omega_c \gg \frac{1}{\tau}$ limit:

$$\begin{aligned} \hat{H}_r^{(2)} = & \frac{g}{2} e^{-\frac{(\omega_a - \omega_c)^2 \tau^2}{2}} \left(e^{-i(\omega_a - \omega_c)t} \hat{a} \hat{\sigma}_+ + e^{+i(\omega_a - \omega_c)t} \hat{a}^\dagger \hat{\sigma}_- \right) + \frac{g^2}{8} \left(\frac{1 - e^{-(\omega_a - \omega_c)^2 \tau^2}}{\omega_a - \omega_c} + \frac{1}{\omega_a + \omega_c} \right) \hat{\sigma}_z \\ & + \frac{g^2}{4} \left(\frac{1 - e^{-(\omega_a - \omega_c)^2 \tau^2}}{\omega_a - \omega_c} + \frac{1}{\omega_a + \omega_c} \right) \hat{a}^\dagger \hat{a} \hat{\sigma}_z \end{aligned} \quad (\text{E4})$$

while the “counter-rotating” terms $\hat{H}_{cr}^{(2)}$ will be exponentially suppressed in the same limit:

$$\begin{aligned} \hat{H}_{cr}^{(2)} = & \frac{g}{2} e^{-\frac{(\omega_a + \omega_c)^2 \tau^2}{2}} \left(e^{i(\omega_a + \omega_c)t} \hat{a}^\dagger \hat{\sigma}_+ + e^{-i(\omega_a + \omega_c)t} \hat{a} \hat{\sigma}_- \right) \\ & + \frac{g^2}{4} \left(e^{-2\omega_c^2 \tau^2} - e^{-(\omega_a^2 + \omega_c^2) \tau^2} \right) \frac{\omega_a}{\omega_a^2 - \omega_c^2} [e^{-2i\omega_c t} \hat{a}^2 \hat{\sigma}_z + e^{2i\omega_c t} \hat{a}^{\dagger 2} \hat{\sigma}_z]. \end{aligned} \quad (\text{E5})$$

In addition to the effective Hamiltonian, TCG also gives rise to effective pseudo-dissipators. In particular, at the second order, we can write $\hat{D}^{(k)}$ as

$$\hat{D}^{(k)} = \hat{D}_r^{(k)} + \hat{D}_{cr}^{(k)} \quad (\text{E6})$$

where the rotating and counter-rotating pseudo-dissipators can be written as

$$\hat{D}_r^{(k)} = \frac{g^2}{2} \cdot \frac{e^{-(\omega_a - \omega_c)^2 \tau^2} - e^{-2(\omega_a - \omega_c)^2 \tau^2}}{\omega_a - \omega_c} e^{-2i(\omega_a - \omega_c)t} \mathcal{D}[\hat{a}^\dagger \hat{\sigma}_-, \hat{a}^\dagger \hat{\sigma}_-] + h.c. \quad (\text{E7})$$

and

$$\begin{aligned} \hat{D}_{cr}^{(k)} = & \frac{g^2}{2} \left(e^{-2\omega_c^2 \tau^2} - e^{-(\omega_a^2 + \omega_c^2) \tau^2} \right) \frac{\omega_c}{\omega_a^2 - \omega_c^2} e^{-2i\omega_c t} \left(\mathcal{D}[\hat{a} \hat{\sigma}_+, \hat{a} \hat{\sigma}_-] + \mathcal{D}[\hat{a} \hat{\sigma}_-, \hat{a} \hat{\sigma}_+] \right) \\ & + \frac{g^2}{2} \left(e^{-2\omega_a^2 \tau^2} - e^{-(\omega_c^2 + \omega_a^2) \tau^2} \right) \frac{\omega_a}{\omega_c^2 - \omega_a^2} e^{-2i\omega_a t} \left(\mathcal{D}[\hat{a} \hat{\sigma}_-, \hat{a}^\dagger \hat{\sigma}_-] + \mathcal{D}[\hat{a}^\dagger \hat{\sigma}_-, \hat{a} \hat{\sigma}_-] \right) \\ & + \frac{g^2}{2} \left(e^{-(\omega_a + \omega_c)^2 \tau^2} - e^{-2(\omega_a + \omega_c)^2 \tau^2} \right) \frac{1}{\omega_a + \omega_c} e^{-2i(\omega_a + \omega_c)t} \mathcal{D}[\hat{a} \hat{\sigma}_-, \hat{a} \hat{\sigma}_-] + h.c. \end{aligned} \quad (\text{E8})$$

respectively.

2. The driven Kerr-parametron

Assuming the numerical values of the parameters presented in Eq.(E9),

$$\begin{aligned}\omega_p/2\pi &= 16\text{GHz}; & \Delta_0/2\pi &= -67\text{MHz}; \\ \beta_0/2\pi &= 200\text{MHz}; & \chi/2\pi &= 68\text{MHz}\end{aligned}\tag{E9}$$

we present all TCG super-operators with coefficients greater than 0.08MHz for a coarse-graining time scale of $\tau = 0.125\text{ns}$. The resulting TCG effective Hamiltonian and dissipators can be written as

$$\hat{H}_{\text{TCG}}(t) = g_{11}\hat{a}^\dagger\hat{a} + g_{22}\hat{a}^{\dagger 2}\hat{a}^2 + g_{33}\hat{a}^{\dagger 3}\hat{a}^3 + (g_{20}\hat{a}^{\dagger 2} + g_{31}\hat{a}^{\dagger 3}\hat{a} + h.c.)\tag{E10}$$

and

$$\begin{aligned}\hat{D}_{\text{TCG}}(t) &= \Gamma_{2,0;0,2}\mathcal{D}[\hat{a}^{\dagger 2}, \hat{a}^2] + \Gamma_{0,2;2,0}\mathcal{D}[\hat{a}^2, \hat{a}^{\dagger 2}] + \left(\Gamma_{2,0;2,0}\mathcal{D}[\hat{a}^{\dagger 2}, \hat{a}^{\dagger 2}] + \Gamma_{2,0;2,2}\mathcal{D}[\hat{a}^{\dagger 2}, \hat{a}^{\dagger 2}\hat{a}^2]\right. \\ &\quad \left.+ \Gamma_{0,2;2,2}\mathcal{D}[\hat{a}^2, \hat{a}^{\dagger 2}\hat{a}^2] + \Gamma_{1,1;2,0}\mathcal{D}[\hat{a}^\dagger\hat{a}, \hat{a}^{\dagger 2}] + \Gamma_{1,1;0,2}\mathcal{D}[\hat{a}^\dagger\hat{a}, \hat{a}^2] + h.c.\right)\end{aligned}\tag{E11}$$

respectively, with

$$\begin{aligned}g_{11} &= \Delta(t) - \frac{2(\beta(t)^2 + \beta(\tau)^2)}{\omega_p} - \frac{4\chi^2}{\omega_p} \\ &\quad + \Delta_0\beta(\tau)^2\left(20\tau^2 + \frac{11}{2\omega_p^2}\right) - \Delta(t)\beta(\tau)^2\left(26\tau^2 + \frac{13}{\omega_p^2}\right) - \left(\frac{\tau^2\Delta_0}{T}\right)^2\Delta(t) + \frac{3}{2}\frac{\beta_0\Delta_0}{T^2\omega_p^4}\beta(t) \\ &\quad + 2\left(\frac{\beta(t)}{\omega_p}\right)^2\Delta(t) - \frac{51}{2}\left(\frac{\beta_0}{T\omega_p^2}\right)^2\chi - \left(6\left(\frac{\beta(t)}{\omega_p}\right)^2 - 4\left(\frac{\beta(\tau)}{\omega_p}\right)^2\right)\chi + 8\left(\frac{\chi}{\omega_p}\right)^2\Delta(t) - \frac{151\chi^3}{6\omega_p^2}\end{aligned}\tag{E12}$$

$$\begin{aligned}g_{22} &= -\frac{\chi}{2} - \frac{17}{4}\frac{\chi^2}{\omega_p} + \Delta_0\beta(\tau)^2\left(16\tau^2 + \frac{26}{\omega_p^2}\right) - \Delta(t)\beta(\tau)^2\left(24\tau^2 + \frac{52}{\omega_p^2}\right) - \left(\frac{\tau^2\Delta_0}{T}\right)^2(3\Delta(t) - 2\chi) \\ &\quad + 7\beta(\tau)^2\tau^2\chi - 9\left(\frac{\beta_0}{T\omega_p^2}\right)^2\chi - \left(4\left(\frac{\beta(t)}{\omega_p}\right)^2 - \frac{137}{4}\left(\frac{\beta(\tau)}{\omega_p}\right)^2\right)\chi + \frac{17}{2}\left(\frac{\chi}{\omega_p}\right)^2\Delta(t) - \frac{199}{4}\frac{\chi^3}{\omega_p^2}\end{aligned}\tag{E13}$$

$$\begin{aligned}g_{33} &= -\frac{17}{18}\frac{\chi^2}{\omega_p} + \Delta_0\beta(\tau)^2\left(4\tau^2 + \frac{17}{2\omega_p^2}\right) - \Delta(t)\beta(\tau)^2\left(6\tau^2 + \frac{17}{\omega_p^2}\right) - \left(\frac{\tau^2\Delta_0}{T}\right)^2\left(\Delta(t) - \frac{5}{2}\chi\right) \\ &\quad + \beta(\tau)^2\left(6\tau^2 + \frac{191}{6\omega_p^2}\right)\chi + \frac{5}{2}\left(\frac{\beta_0}{T\omega_p^2}\right)^2\chi + \frac{17}{9}\left(\frac{\chi}{\omega_p}\right)^2\Delta(t) - \frac{233\chi^3}{9\omega_p^2}\end{aligned}\tag{E14}$$

$$\begin{aligned}
g_{20} = & \beta(t) + \frac{5\chi}{2\omega_p}\beta(t) - \frac{17}{4}\frac{\chi^2}{\omega_p} + i\left(\beta(\tau)\tau\left(\Delta_0 - \frac{\chi}{2}\right) - \frac{9\beta_0\chi}{8T\omega_p^2}\right) \\
& + \left(16\Delta_0 - 17\beta(t)\right)\beta(\tau)^2\tau^2 - 24\Delta(t)\beta(\tau)^2\tau^2 - (7\beta(t) - 4\beta_0)\left(\frac{\tau^2\Delta_0}{T}\right)^2 - 3\Delta(t)\left(\frac{\tau^2\Delta_0}{T}\right)^2 \\
& + \frac{15}{2}\beta(\tau^2)\tau^2\chi - 2\frac{(3\beta_0 - \Delta_0)\Delta_0\chi\tau^4}{T^2} + \left(\frac{11\beta_0}{2T\omega_p^2}\right)^2\beta(t) - \frac{53}{6}\left(\frac{\beta_0}{T\omega_p^2}\right)^2\chi - \frac{3\beta_0\Delta_0\chi}{16(T\omega_p^2)^2} \\
& - \frac{7\beta(t)^3}{\omega_p^2} - \left(\frac{\beta_0\tau}{T\omega_p}\right)^2(74\beta(t) + 52\Delta(t) - 26\Delta_0) + \frac{197}{12}\left(\frac{\chi}{\omega_p}\right)^2\beta(t) + \frac{17}{2}\left(\frac{\chi}{\omega_p}\right)^2\Delta(t) - \frac{199\chi^3}{4\omega_p^2} \\
& + \frac{\frac{559}{16}\beta(\tau)^2 - \frac{83}{48}\beta(\tau)\frac{\tau}{T}\Delta_0 - 4\beta(t)^2 - \frac{11}{4}\beta(t)\Delta(t)}{\omega_p^2}\chi \\
& + i\left[\left(2\beta(t)^2\tau^2 + \frac{7(\beta(t)^2 + \beta(\tau)^2)}{2\omega_p^2}\right)\frac{\beta_0}{T\omega_p} - \frac{\chi\left(\frac{\beta_0}{6} - \frac{15}{8}\Delta_0\right)}{T\omega_p^3}\beta(t) + \frac{\beta_0\chi}{T\omega_p^3}\left(\frac{\Delta_0}{4} - \frac{167\chi}{24}\right) \right. \\
& \left. + \frac{5\beta_0\Delta_0\chi\tau^2}{2T\omega_p} - \frac{5\beta_0\tau^2\chi^2}{T\omega_p}\right]
\end{aligned} \tag{E15}$$

$$\begin{aligned}
g_{31} = & \frac{5\chi}{3\omega_p}\beta(t) - i\left[\tau\beta(\tau)\chi + \frac{3\beta_0\chi}{4T\omega_p^2}\right] \\
& - 3\tau^2\left(4\beta(\tau)^2 + 3\left(\frac{\tau\Delta_0}{T}\right)^2\right)\beta(t) - 9\left(\frac{\beta_0}{T\omega_p^2}\right)^2\beta(t) - 52\left(\frac{\beta(\tau)}{\omega_p}\right)^2\beta(t) + \frac{\beta_0\Delta_0\tau^4}{T^2}(6\Delta_0 - 5\chi) \\
& - \frac{11\chi}{6\omega_p^2}\Delta(t)\beta(t) - \frac{\beta_0\Delta_0\chi}{8T^2\omega_p^2}\left(27\tau^2 + \frac{1}{\omega_p^2}\right) + \frac{119}{6}\left(\frac{\chi}{\omega_p}\right)^2\beta(t) \\
& + i\left[\frac{5\Delta_0\chi}{4T\omega_p^3}\beta(t) - \frac{9\beta_0}{T\omega_p^3}\left(\beta(t)^2 + \beta(\tau)^2\right) + \frac{\beta_0\Delta_0\chi}{6T\omega_p^3} - \frac{50\beta_0\chi^2}{3T\omega_p^3} + \frac{5\beta_0\Delta_0\chi\tau^2}{3T\omega_p} - \frac{49\beta_0\tau^2\chi^2}{3T\omega_p}\right]
\end{aligned} \tag{E16}$$

$$\begin{aligned}
\Gamma_{2,0;0,2} = \Gamma_{0,2;2,0} = & \frac{\beta_0}{T}\left(2\tau^2 + \frac{1}{2\omega_p^2}\right)\beta(t); \quad \Gamma_{2,0;2,0} = \frac{2\tau^2\beta_0}{T}\beta(t); \quad \Gamma_{2,0;2,2} = \Gamma_{0,2;2,2} = -\frac{\tau\chi}{2}\beta(\tau) \\
\Gamma_{1,1;2,0} = & -\frac{45\beta_0}{2T\omega_p^3}\left(\beta(t)^2 + \beta(\tau)^2\right) - \frac{2\tau^2\beta_0}{T\omega_p}\left(3\beta(t)^2 + 2\beta(\tau)^2\right) - \frac{5}{2}\left(2\beta(t) - \beta_0\right)\frac{\tau^2\Delta_0\chi}{T\omega_p} \\
& - \frac{2\Delta_0\chi}{T\omega_p^3}\beta(t) - \frac{19\beta_0\chi^2}{3T\omega_p^3} - \frac{4\tau\chi^2}{\omega_p}\beta(\tau) + i\left[3\tau^2\left(\left(\frac{\tau\Delta_0}{T}\right)^2 - 4\beta(\tau)^2\right)\beta(t) - 4\left(\frac{\tau^2\Delta_0}{T}\right)^2\beta_0 \right. \\
& + 2\tau\beta(\tau)\left(\frac{\tau^2\Delta_0}{T}\right)\chi + \frac{183}{4}\left(\frac{\beta_0}{T\omega_p^2}\right)^2\beta(t) - 17\frac{\beta(t)^3}{\omega_p^2} - 39\left(\frac{\beta(\tau)}{\omega_p}\right)^2\beta(t) - \frac{2\beta(t)\Delta(t)}{\omega_p^2}\chi \\
& \left. + \frac{13\tau^2\beta_0\Delta_0\chi}{4T^2\omega_p^2} - \left(\frac{\chi}{\omega_p}\right)^2\beta(t)\right]
\end{aligned} \tag{E17}$$

$$\begin{aligned}
\Gamma_{1,1;0,2} = & -\frac{33\beta_0}{2T\omega_p^3}\left(\beta(t)^2 + \beta(\tau)^2\right) - \frac{2\tau^2\beta_0}{T\omega_p}\left(3\beta(t)^2 + 2\beta(\tau)^2\right) - \frac{5}{2}\left(2\beta(t) - \beta_0\right)\frac{\tau^2\Delta_0\chi}{T\omega_p} \\
& - \frac{2\Delta_0\chi}{T\omega_p^3}\beta(t) - \frac{25\beta_0\chi^2}{6T\omega_p^3} - \frac{4\tau\chi^2}{\omega_p}\beta(\tau) + i\left[3\tau^2\left(\left(\frac{\tau\Delta_0}{T}\right)^2 - 4\beta(\tau)^2\right)\beta(t) \right. \\
& - \frac{159}{4}\left(\frac{\beta_0}{T\omega_p^2}\right)^2\beta(t) + 17\frac{\beta(t)^3}{\omega_p^2} + 63\left(\frac{\beta(\tau)}{\omega_p}\right)^2\beta(t) + \frac{2\beta(t)\Delta(t)}{\omega_p^2}\chi - \frac{\tau^2\beta_0\Delta_0\chi}{T^2\omega_p^2} \\
& \left. + \left(\frac{\chi}{\omega_p}\right)^2\beta(t)\right]
\end{aligned}$$

Although we assumed particular numerical values of the parameters for truncating the TCG master equation, the formulas above can be used as good approximations as long as the coarse-graining time scale is in the range $\frac{1}{|\beta(t)|}, \frac{1}{|\Delta(t)|}, \frac{1}{|\chi|} \gg \tau \gg \frac{1}{\omega_p}$. Here the first inequality ensures the validity of the TCG perturbative expansion regardless of the frequencies of the terms involved, whereas the second inequality allows us to ignore the highly oscillatory terms in the TCG master equation.

3. The driven Duffing oscillator

The fourth-order effective TCG Hamiltonian can be written as

$$\hat{H}_{\text{TCG}}^{(4)} \equiv K_1^{(4)} \hat{a}^\dagger \hat{a} + K_2^{(4)} \hat{a}^{\dagger 2} \hat{a}^2 + K_3^{(4)} \hat{a}^{\dagger 3} \hat{a}^3 + K_4^{(4)} \hat{a}^{\dagger 4} \hat{a}^4 + K_5^{(4)} \hat{a}^{\dagger 5} \hat{a}^5 \quad (\text{E18})$$

The approximate expressions for the coefficients $K_n^{(4)}$ are given as,

$$\begin{aligned} K_1^{(4)} \approx & \frac{g_4^2}{\omega} (-58 + 625|\Pi|^2 + 531|\Pi|^4) \\ & + \frac{g_4^2}{\omega^2} \delta (12 + 907|\Pi|^2 + 665|\Pi|^4) \\ & + \frac{g_4^3}{\omega^2} (573 + 13815|\Pi|^2 + 43258|\Pi|^4 + 21832|\Pi|^6) \\ & + \frac{g_4^4}{\omega^3} (-7834 + 246852|\Pi|^2 + 2101485|\Pi|^4 \\ & + 3034553|\Pi|^6 + 1079394|\Pi|^8) \\ & + \frac{g_4^3 \delta}{\omega^3} (-229 + 35144|\Pi|^2 + 106765|\Pi|^4 + 51312|\Pi|^6) \\ & + \frac{g_4^2 \delta^2}{\omega^3} (-2 + 855|\Pi|^2 + 645|\Pi|^4) \end{aligned} \quad (\text{E19})$$

Furthermore, the exact analytical formulas for the coefficients can be written as follows,

$$\begin{aligned} K_2^{(4)} \approx & \frac{g_4^2}{\omega} (-61 + 312|\Pi|^2) + \frac{g_4^2}{\omega^2} \delta (12 + 453|\Pi|^2) \\ & + \frac{g_4^3}{\omega^2} (1007 + 17919|\Pi|^2 + 21629|\Pi|^4) \\ & + \frac{g_4^4}{\omega^3} (-20529 + 655974|\Pi|^2 \\ & + 2723568|\Pi|^4 + 1517277|\Pi|^6) \\ & + \frac{g_4^3 \delta}{\omega^3} (-403 + 46418|\Pi|^2 + 53383|\Pi|^4) \\ & + \frac{g_4^2 \delta^2}{\omega^3} (-2 + 427|\Pi|^2) \end{aligned} \quad (\text{E20})$$

$$\begin{aligned} K_3^{(4)} \approx & -\frac{g_4^2}{\omega} \cdot 14 + \frac{g_4^2}{\omega^2} \delta \cdot 3 + \frac{g_4^3}{\omega^2} (480 + 3982|\Pi|^2) \\ & + \frac{g_4^4}{\omega^3} (-15964 + 355032|\Pi|^2 + 605237|\Pi|^4) \\ & + \frac{g_4^3 \delta}{\omega^3} (-192 + 10315|\Pi|^2) - \frac{g_4^2 \delta^2}{\omega^3} \end{aligned} \quad (\text{E21})$$

$$K_4^{(4)} \approx \frac{g_4^3}{\omega^2} \cdot 60 + \frac{g_4^4}{\omega^3} (-4276 + 44379|\Pi|^2) - \frac{g_4^3 \delta}{\omega^3} \cdot 24 \quad (\text{E22})$$

$$K_5^{(4)} \approx -\frac{g_4^4}{\omega^3} \cdot 342 \quad (\text{E23})$$

$$\begin{aligned} K_1^{(4)} &= \frac{g_4^2}{\omega} \left(-\frac{288}{5} + \frac{240448}{385}|\Pi|^2 + \frac{29232}{55}|\Pi|^4 \right) + \frac{g_4^2}{\omega^2} \delta \left(\frac{288}{25} + \frac{403234112}{444675}|\Pi|^2 + \frac{2011248}{3025}|\Pi|^4 \right) \\ &+ \frac{g_4^3}{\omega^2} \left(\frac{14328}{25} + \frac{1228651264}{88935}|\Pi|^2 + \frac{6411937792}{148225}|\Pi|^4 + \frac{5904212871552}{270438025}|\Pi|^6 \right) \\ &+ \frac{g_4^4}{\omega^3} \left(-\frac{39168}{5} + \frac{8297258809465088}{33612242125}|\Pi|^2 + \frac{215864405353424}{102719925}|\Pi|^4 \right. \\ &\quad \left. + \frac{4629035610176328876032}{1525442189145875}|\Pi|^6 + \frac{16321516698840554304}{15121001291825}|\Pi|^8 \right) \\ &+ \frac{g_4^3 \delta}{\omega^3} \left(-\frac{28656}{125} + \frac{3609939830272}{102719925}|\Pi|^2 + \frac{18278157853696}{171199875}|\Pi|^4 + \frac{228204470088870144}{4447353321125}|\Pi|^6 \right) \\ &+ \frac{g_4^2 \delta^2}{\omega^3} \left(-\frac{288}{125} + \frac{439080132928}{513599625}|\Pi|^2 + \frac{107346672}{166375}|\Pi|^4 \right) \end{aligned} \quad (\text{E24})$$

$$\begin{aligned} K_2^{(4)} &= \frac{g_4^2}{\omega} \left(-\frac{306}{5} + \frac{120224}{385}|\Pi|^2 \right) + \frac{g_4^2}{\omega^2} \delta \left(\frac{306}{25} + \frac{201617056}{444675}|\Pi|^2 \right) \\ &+ \frac{g_4^3}{\omega^2} \left(\frac{25164}{25} + \frac{531220656}{29645}|\Pi|^2 + \frac{3205968896}{148225}|\Pi|^4 \right) \\ &+ \frac{g_4^4}{\omega^3} \left(-\frac{513234}{25} + \frac{22048747044860544}{33612242125}|\Pi|^2 + \frac{2825906273812}{1037575}|\Pi|^4 \right. \\ &\quad \left. + \frac{2314517805088164438016}{1525442189145875}|\Pi|^6 \right) \\ &+ \frac{g_4^3 \delta}{\omega^3} \left(-\frac{50328}{125} + \frac{529778123296}{11413325}|\Pi|^2 + \frac{9139078926848}{171199875}|\Pi|^4 \right) \\ &+ \frac{g_4^2 \delta^2}{\omega^3} \left(-\frac{306}{125} + \frac{219540066464}{513599625}|\Pi|^2 \right) \end{aligned} \quad (\text{E25})$$

$$\begin{aligned} K_3^{(4)} &= -\frac{g_4^2}{\omega} \cdot \frac{68}{5} + \frac{g_4^2}{\omega^2} \delta \cdot \frac{68}{25} + \frac{g_4^3}{\omega^2} \left(480 + \frac{354147104}{88935}|\Pi|^2 \right) \\ &+ \frac{g_4^4}{\omega^3} \left(-\frac{399092}{25} + \frac{286401882242048}{806693811}|\Pi|^2 + \frac{5651812547624}{9338175}|\Pi|^4 \right) \\ &+ \frac{g_4^3 \delta}{\omega^3} \left(-192 + \frac{1059556246592}{102719925}|\Pi|^2 \right) - \frac{g_4^2 \delta^2}{\omega^3} \frac{68}{125} \end{aligned} \quad (\text{E26})$$

$$K_4^{(4)} = \frac{g_4^3}{\omega^2} \cdot 60 + \frac{g_4^4}{\omega^3} \left(-\frac{21378}{5} + \frac{35800235280256}{806693811}|\Pi|^2 \right) - \frac{g_4^3 \delta}{\omega^3} \cdot 24 \quad (\text{E27})$$

$$K_5^{(4)} = -\frac{g_4^4}{\omega^3} \cdot \frac{42756}{125} \quad (\text{E28})$$

The TCG dissipators in $D_{\text{TCG}}(t)$ start to appear at the third-order in the perturbative expansion. Keeping the

lowest-order contributions, we have

$$\begin{aligned}
\hat{D}_{\text{TCG}}^{(3)} \equiv & \Gamma_{2,1;0,1}^{(3)} \mathcal{D}[\hat{a}^{\dagger 2} \hat{a}, \hat{a}] + \Gamma_{0,1;2,1}^{(3)} \mathcal{D}[\hat{a}, \hat{a}^{\dagger 2} \hat{a}] \\
& + \Gamma_{2,2;1,1}^{(3)} \mathcal{D}[\hat{a}^{\dagger 2} \hat{a}^2, \hat{a}^{\dagger} \hat{a}] + \Gamma_{2,3;1,0}^{(3)} \mathcal{D}[\hat{a}^{\dagger 2} \hat{a}^3, \hat{a}^{\dagger}] \\
& + \Gamma_{1,0;2,3}^{(3)} \mathcal{D}[\hat{a}^{\dagger}, \hat{a}^{\dagger 2} \hat{a}^3] + \Gamma_{1,3;2,0}^{(3)} \mathcal{D}[\hat{a}^{\dagger} \hat{a}^3, \hat{a}^{\dagger 2}] \\
& + \Gamma_{2,0;1,3}^{(3)} \mathcal{D}[\hat{a}^{\dagger 2}, \hat{a}^{\dagger} \hat{a}^3] + \Gamma_{3,3;1,1}^{(3)} \mathcal{D}[\hat{a}^{\dagger 3} \hat{a}^3, \hat{a}^{\dagger} \hat{a}] \\
& + \Gamma_{4,2;0,2}^{(3)} \mathcal{D}[\hat{a}^{\dagger 4} \hat{a}^2, \hat{a}^2] + \Gamma_{0,2;4,2}^{(3)} \mathcal{D}[\hat{a}^2, \hat{a}^{\dagger 4} \hat{a}^2] \\
& + \Gamma_{4,1;0,3}^{(3)} \mathcal{D}[\hat{a}^{\dagger 4} \hat{a}, \hat{a}^3] + \Gamma_{0,3;4,1}^{(3)} \mathcal{D}[\hat{a}^3, \hat{a}^{\dagger 4} \hat{a}] \\
& + \Gamma_{3,4;1,0}^{(3)} \mathcal{D}[\hat{a}^{\dagger 3} \hat{a}^4, \hat{a}^{\dagger}] + \Gamma_{1,0;3,4}^{(3)} \mathcal{D}[\hat{a}^{\dagger}, \hat{a}^{\dagger 3} \hat{a}^4] \\
& + \Gamma_{3,2;1,2}^{(3)} \mathcal{D}[\hat{a}^{\dagger 3} \hat{a}^2, \hat{a}^{\dagger} \hat{a}^2] + \Gamma_{1,2;3,2}^{(3)} \mathcal{D}[\hat{a}^{\dagger} \hat{a}^2, \hat{a}^{\dagger 3} \hat{a}^2] \\
& + \Gamma_{3,3;2,2}^{(3)} \mathcal{D}[\hat{a}^{\dagger 3} \hat{a}^3, \hat{a}^{\dagger 2} \hat{a}^2] + \Gamma_{4,4;1,1}^{(3)} \mathcal{D}[\hat{a}^{\dagger 4} \hat{a}^4, \hat{a}^{\dagger} \hat{a}] \\
& + \Gamma_{4,2;1,3}^{(3)} \mathcal{D}[\hat{a}^{\dagger 4} \hat{a}^2, \hat{a}^{\dagger} \hat{a}^3] + \Gamma_{1,3;4,2}^{(3)} \mathcal{D}[\hat{a}^{\dagger} \hat{a}^3, \hat{a}^{\dagger 4} \hat{a}^2] \\
& + \Gamma_{5,1;0,4}^{(3)} \mathcal{D}[\hat{a}^{\dagger 5} \hat{a}, \hat{a}^4] + \Gamma_{0,4;5,1}^{(3)} \mathcal{D}[\hat{a}^4, \hat{a}^{\dagger 5} \hat{a}] \\
& + \Gamma_{3,4;2,1}^{(3)} \mathcal{D}[\hat{a}^{\dagger 3} \hat{a}^4, \hat{a}^{\dagger 2} \hat{a}] + \Gamma_{2,1;3,4}^{(3)} \mathcal{D}[\hat{a}^{\dagger 2} \hat{a}, \hat{a}^{\dagger 3} \hat{a}^4] \\
& + \Gamma_{5,2;0,3}^{(3)} \mathcal{D}[\hat{a}^{\dagger 5} \hat{a}^2, \hat{a}^3] + \Gamma_{0,3;5,2}^{(3)} \mathcal{D}[\hat{a}^3, \hat{a}^{\dagger 5} \hat{a}^2] \\
& + \Gamma_{3,5;2,0}^{(3)} \mathcal{D}[\hat{a}^{\dagger 3} \hat{a}^5, \hat{a}^{\dagger 2}] + \Gamma_{2,0;3,5}^{(3)} \mathcal{D}[\hat{a}^{\dagger 2}, \hat{a}^{\dagger 3} \hat{a}^5] \\
& + \Gamma_{4,4;2,2}^{(3)} \mathcal{D}[\hat{a}^{\dagger 4} \hat{a}^4, \hat{a}^{\dagger 2} \hat{a}^2] + \Gamma_{6,2;0,4}^{(3)} \mathcal{D}[\hat{a}^{\dagger 6} \hat{a}^2, \hat{a}^4] \\
& + \Gamma_{0,4;6,2}^{(3)} \mathcal{D}[\hat{a}^4, \hat{a}^{\dagger 6} \hat{a}^2] + \Gamma_{3,5;3,1}^{(3)} \mathcal{D}[\hat{a}^{\dagger 3} \hat{a}^5, \hat{a}^{\dagger 3} \hat{a}^1] \\
& + \Gamma_{3,1;3,5}^{(3)} \mathcal{D}[\hat{a}^{\dagger 3} \hat{a}^1, \hat{a}^{\dagger 3} \hat{a}^5] + h.c.
\end{aligned} \tag{E29}$$

The analytical expressions for the coefficients $\Gamma_{\alpha,\beta;\gamma,\rho}^{(3)}$ are given below.

$$\begin{aligned}
\Gamma_{2,1;0,1}^{(3)} = & -i \frac{g_4^3}{\omega^2} \left(\frac{421632}{121} |\Pi|^2 + \frac{1475712}{121} |\Pi|^4 + \frac{131946344448}{10817521} |\Pi|^6 + \frac{37726754304}{10817521} |\Pi|^8 \right) \\
& - i \frac{g_4^2 \delta}{\omega^2} \left(\frac{35136}{121} |\Pi|^2 + \frac{52704}{121} |\Pi|^4 + \frac{1571948096}{10817521} |\Pi|^6 \right)
\end{aligned} \tag{E30}$$

$$\begin{aligned}
\Gamma_{0,1;2,1}^{(3)} = & i \frac{g_4^3}{\omega^2} \left(\frac{843264}{121} |\Pi|^2 + \frac{2108160}{121} |\Pi|^4 + \frac{150809721600}{10817521} |\Pi|^6 + \frac{37726754304}{10817521} |\Pi|^8 \right) \\
& + i \frac{g_4^2 \delta}{\omega^2} \left(\frac{35136}{121} |\Pi|^2 + \frac{52704}{121} |\Pi|^4 + \frac{1571948096}{10817521} |\Pi|^6 \right)
\end{aligned} \tag{E31}$$

$$\begin{aligned}
\Gamma_{2,2;1,1}^{(3)} = & -i g_4^3 (1 + |\Pi|^2) (5184\tau^2 + 17280\tau^2 |\Pi|^2 + (\frac{48}{\omega^2} + 13824\tau^2) |\Pi|^4) \\
& - i g_4^2 \delta (936\tau^2 + 2592\tau^2 |\Pi|^2 + (1728\tau^2 + \frac{2}{\omega^2}) |\Pi|^4) - i g_4 \delta^2 \tau^2 (54 + 72 |\Pi|^2) - i \delta^3 \tau^2
\end{aligned} \tag{E32}$$

$$\Gamma_{2,3;1,0}^{(3)} = -i \frac{g_4^3}{\omega^2} \left(\frac{737856}{121} |\Pi|^2 + \frac{1264896}{121} |\Pi|^4 + \frac{47126011008}{10817521} |\Pi|^6 \right) - i \frac{\delta_4^2 \delta}{\omega^2} \frac{17568}{121} (|\Pi|^2 + |\Pi|^4) \tag{E33}$$

$$\Gamma_{1,0;2,3}^{(3)} = i \frac{g_4^3}{\omega^2} \left(\frac{527040}{121} |\Pi|^2 + \frac{1054080}{121} |\Pi|^4 + \frac{47126011008}{10817521} |\Pi|^6 \right) + i \frac{g_4^2 \delta}{\omega^2} \frac{17568}{121} (|\Pi|^2 + |\Pi|^4) \tag{E34}$$

$$\begin{aligned}\Gamma_{1,3;2,0}^{(3)} = & -i\frac{g_4^3}{\omega^2}\left(\frac{108}{5} + \frac{2088}{25}|\Pi|^2 + \frac{1284408}{3025}|\Pi|^4 + \frac{763344}{3025}|\Pi|^6\right) \\ & -i\frac{g_4^2\delta}{\omega^2}\left(\frac{3}{5} + \frac{48}{25}|\Pi|^2 + \frac{31806}{3025}|\Pi|^4\right)\end{aligned}\quad (\text{E35})$$

$$\Gamma_{2,0;1,3}^{(3)} = i\frac{g_4^3}{\omega^2}\left(\frac{36}{5} + \frac{936}{25}|\Pi|^2 + \frac{521064}{3025}|\Pi|^4 + \frac{763344}{3025}|\Pi|^6\right) + i\frac{g_4^2\delta}{\omega^2}\left(\frac{3}{5} + \frac{48}{25}|\Pi|^2 + \frac{31806}{3025}|\Pi|^4\right) \quad (\text{E36})$$

$$\begin{aligned}\Gamma_{3,3;1,1}^{(3)} = & -ig_4^3(3456\tau^2 + 10368\tau^2|\Pi|^2 + (\frac{12}{\omega^2} + 6912\tau^2)|\Pi|^4) - ig_4^2\delta(432\tau^2 + 576\tau^2|\Pi|^2) \\ & - i12g_4\delta^2\tau^2\end{aligned}\quad (\text{E37})$$

$$\Gamma_{4,2;0,2}^{(3)} = -i\frac{g_4^3}{\omega^2}\left(\frac{198}{25} + \frac{648}{25}|\Pi|^2 + \frac{225684}{3025}|\Pi|^4\right) - i\frac{g_4^2\delta}{\omega^2}\frac{6}{25}(1 + 2|\Pi|^2) \quad (\text{E38})$$

$$\Gamma_{0,2;4,2}^{(3)} = i\frac{g_4^3}{\omega^2}\left(\frac{342}{25} + \frac{936}{25}|\Pi|^2 + \frac{225684}{3025}|\Pi|^4\right) + i\frac{g_4^2\delta}{\omega^2}\frac{6}{25}(1 + 2|\Pi|^2) \quad (\text{E39})$$

$$\Gamma_{4,1;0,3}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{3712}{1323}(|\Pi|^2 + 2|\Pi|^4) - i\frac{g_4^2\delta}{\omega^2}\frac{928}{3969}|\Pi|^2 \quad (\text{E40})$$

$$\Gamma_{0,3;4,1}^{(3)} = i\frac{g_4^3}{\omega^2}\frac{7424}{1323}(2|\Pi|^2 + |\Pi|^4) + i\frac{g_4^2\delta}{\omega^2}\frac{928}{3969}|\Pi|^2 \quad (\text{E41})$$

$$\Gamma_{3,4;1,0}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{105408}{121}(|\Pi|^2 + |\Pi|^4); \quad \Gamma_{1,0;3,4}^{(3)} = i\frac{g_4^3}{\omega^2}\frac{105408}{121}(|\Pi|^2 + |\Pi|^4) \quad (\text{E42})$$

$$\Gamma_{3,2;1,2}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{527040}{121}(|\Pi|^2 + |\Pi|^4) - i\frac{g_4^2\delta}{\omega^2}\frac{17568}{121}|\Pi|^2 \quad (\text{E43})$$

$$\Gamma_{1,2;3,2}^{(3)} = i\frac{g_4^3}{\omega^2}\frac{105408}{121}(7|\Pi|^2 + 5|\Pi|^4) + i\frac{g_4^2\delta}{\omega^2}\frac{17568}{121}|\Pi|^2 \quad (\text{E44})$$

$$\Gamma_{3,3;2,2}^{(3)} = -i1728g_4^3\tau^2(1 + |\Pi|^2) - i72g_4^2\delta\tau^2; \quad \Gamma_{4,4;1,1}^{(3)} = -i432g_4^3\tau^2(1 + 2|\Pi|^2) - i36g_4^2\delta\tau^2 \quad (\text{E45})$$

$$\Gamma_{4,2;1,3}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{12}{25}(11 + 14|\Pi|^2) - i\frac{4}{25}\frac{g_4^2\delta}{\omega^2}; \quad \Gamma_{1,3;4,2}^{(3)} = i\frac{g_4^3}{\omega^2}\frac{12}{25}(19 + 14|\Pi|^2) + i\frac{g_4^2\delta}{\omega^2}\frac{4}{25} \quad (\text{E46})$$

$$\Gamma_{5,1;0,4}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{3}{100}(1 + 2|\Pi|^2) - i\frac{g_4^2\delta}{\omega^2}\frac{1}{400}; \quad \Gamma_{0,4;5,1}^{(3)} = i\frac{g_4^3}{\omega^2}\frac{3}{100}(5 + 2|\Pi|^2) + i\frac{g_4^2\delta}{\omega^2}\frac{1}{400} \quad (\text{E47})$$

$$\Gamma_{3,4;2,1}^{(3)} = -\Gamma_{2,1;3,4}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{105408}{121}|\Pi|^2; \quad \Gamma_{5,2;0,3}^{(3)} = -\Gamma_{0,3;5,2}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{1856}{1323}|\Pi|^2 \quad (\text{E48})$$

$$\Gamma_{3,5;2,0}^{(3)} = -\Gamma_{2,0;3,5}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{36}{25}(1 + 2|\Pi|^2); \quad \Gamma_{4,4;2,2}^{(3)} = -i216g_4^3\tau^2 \quad (\text{E49})$$

$$\Gamma_{6,2;0,4}^{(3)} = -\Gamma_{0,4;6,2}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{3}{200}; \quad \Gamma_{3,5;3,1}^{(3)} = -\Gamma_{3,1;3,5}^{(3)} = -i\frac{g_4^3}{\omega^2}\frac{24}{25} \quad (\text{E50})$$

In particular, we notice that the coefficients of the pseudo-dissipators with excitation-conserving operators on both sides of the density matrix (e.g. $\Gamma_{2,2;1,1}^{(3)}$, $\Gamma_{3,3;1,1}^{(3)}$, $\Gamma_{3,3;2,2}^{(3)}$, $\Gamma_{4,4;1,1}^{(3)}$, and $\Gamma_{4,4;2,2}^{(3)}$) are explicitly dependent on the coarse-graining time scale τ , whereas all the other coefficients are independent of τ . And for the same reason explained in

the discussion of the parametron in the main text, we expect the analytical expression of the TCG master equation above to be accurate as long as one works in the regime where $\frac{1}{g_4} \gg \tau \gg \frac{1}{\omega}$.

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