

Cavity-coupled double-quantum dot at finite bias: Analogy with lasers and beyond

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We present a theoretical and experimental study of photonic and electronic transport properties of a voltage biased InAs semiconductor double quantum dot (DQD) that is dipole coupled to a superconducting transmission line resonator. We obtain the master equation for the reduced density matrix of the coupled system of cavity photons and DQD electrons accounting systematically for both the presence of phonons and the effect of leads at finite voltage bias. We subsequently derive analytical expressions for transmission, phase response, photon number, and the nonequilibrium steady-state electron current. We show that the coupled system under finite bias realizes an unconventional version of a single-atom laser and analyze the spectrum and the statistics of the photon flux leaving the cavity. In the transmission mode, the system behaves as a saturable single-atom amplifier for the incoming photon flux. Finally, we show that the back action of the photon emission on the steady-state current can be substantial. Our analytical results are compared to exact master equation results establishing regimes of validity of various analytical models. We compare our findings to available experimental measurements.

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I. INTRODUCTION

Hybrid light-matter systems offer a unique platform to study nonequilibrium many-body phenomena. Here we study a hybrid system composed of a voltage-biased double quantum dot (DQD) coupled to a superconducting transmission line resonator [1–5]. From the perspective of cavity QED physics which studies fundamental light-matter interaction between a quantum emitter and the radiation field of an electromagnetic resonator, here we encounter a situation where the electronic excitations constituting the emitter cannot be treated as an isolated system. Many-body effects that derive from the hybridization of the DQD electrons with the lead electrons can lead to novel phenomena at the interface of quantum impurity physics and quantum optics. In particular, such a setting provides an accurate probe to study the nonequilibrium dynamics of standard quantum impurity systems. A case in point is recent theoretical [6] and subsequent experimental work [7] that studied the optical signatures of the quench dynamics of a quantum impurity in the Kondo regime. More recent work [8] addressed a strongly (optically) driven quantum impurity beyond the linear response regime and demonstrated the formation of a new quantum-correlated state characterized by the emergence of a secondary spin-screening cloud. Similar quantum dot systems [9,10] coupled to a cavity have been used to investigate electron correlations [11] and the role of dephasing and relaxation [12,13].

Here we study a cavity coupled to a quantum impurity through which a steady-state current is driven. The quantum impurity is a DQD defined on a nanowire through electrostatic gating and held under a static source-drain voltage. Such a system acts as an amplifier for an incident microwave signal. Photon emission from this system has recently been experimentally analyzed [4] showing a gain as large as 15 in the cavity transmission. Earlier theoretical work has addressed statistics of photon emission [14–17] and intracavity photon number [18] but did not touch on the interplay of phonon and photon-induced interdot tunneling processes, which experiments of Ref. [4] indicate to play an important

role in determining the gain and the bandwidth of this cavity-DQD quantum amplifier. A systematic analysis assessing the role of photonic and phononic quantum environment in the nonequilibrium dynamics of this hybrid system has been missing and is one of the main goals of our paper. Pinning down the microscopic origin of the various relaxation and pure dephasing channels can provide a handle on engineering the various couplings to the environment to access the strong-coupling regime crucial for building highly efficient novel amplifiers for quantum microwave signals.

The cavity-coupled DQD system studied here is in some ways analogous to a single-atom laser where the cavity is formed by a superconducting transmission line resonator, the gain medium is formed by the voltage-biased DQD, and pumping is achieved by current driven through the DQD. However, as we will show below, the intricate interplay between phonon and photon-assisted tunneling processes lead to a setting where electron and photon transport can no longer be treated separately. From this perspective this system is rather similar to a quantum cascade laser [19,20] where the active region is composed of a DQD structure. We first derive the master equation for the reduced density matrix of the DQD-cavity system in the sequential tunneling regime with one extra electron injected through the leads. For this we start with a microscopic description of the coupling to the phonon bath that has recently been experimentally studied for InAs nanowire QDs [21]. We derive analytical expressions for transmission, phase response, thresholds, lasing frequencies, photon number, and steady-state electron current. In particular, we show that the nonequilibrium optical susceptibility of the cavity-DQD amplifier features significant renormalization of the gain peak and bandwidth with respect to a bare current-driven DQD. Next we investigate the contributions of photon and phonon emission on the nonequilibrium current driven through the DQD. With recent progress in the measurement of photon correlation functions in superconducting circuits [17,22,23], it is now possible to analyze the statistics of emitted microwave photons. With this in mind, we compute the spectrum and the second order correlation function of photons leaving the cavity.

Analytical results are benchmarked against exact numerics and when applicable this is compared to existing experimental measurements.

The paper is organized as follows. In Sec. II we describe the system studied in recent experiments [4]. We derive the full master equation for a voltage-biased DQD coupled to a single-mode cavity from a microscopic model. In Sec. III we derive an effective two-level model for the combined system that allows us to analyze it in the framework of a single-atom laser. We compute the laser threshold, frequency, spectrum, photon number, and second order correlation function of the photon flux leaving the cavity. We subsequently compute the transmission and the phase response below the semiclassical laser threshold and compare to experimental measurements to establish the experimental regime of parameters. In Sec. IV we calculate the nonequilibrium steady-state current through the DQD and compare to experimental measurements. We devote Sec. V to the comparison of our analytical results with experimental measurements. In this section we also describe our findings on the emission spectrum and the second order correlation function of the photon flux leaving the cavity and its dependence on experimentally tunable parameters. The paper is summarized with an outlook in Sec. VI. Details of some calculations are delegated to the Appendices. Appendix A deals with the electron transport and treatment of electron leads as a bath for the cavity-coupled DQD system. In Appendix B, we derive the equations of motion for our system and derive the various quantities of experimental and theoretical interest.

II. MASTER EQUATION OF THE DQD-CAVITY SYSTEM

As schematically depicted in Fig. 1 we consider a double quantum dot (DQD) that is connected to fermionic reservoirs (FR, chemical potentials μ_L and μ_R , $V_{SD} = \mu_L - \mu_R$), dipole coupled (vacuum Rabi frequency g) to a transmission line resonator (R). The DQD is defined on an InAs nanowire (NW) and is coupled to phonon baths (PHON) of the substrate and the nanowire. We start with a microscopic description of the phonon-photon-DQD-FR

$$H = H_{\text{DQD}} + H_R + H_{\text{DQD-R}} + H_{\text{PHON}} + H_{\text{rmDQD-PHON}} + H_{\text{FR}} + H_{\text{FR-DQD}}. \quad (1)$$

Leaving out the parts involving leads, the remaining Hamiltonian can be written as

$$H_{\text{DQD,R,PH}} = \int_{-\infty}^{\infty} dx \Psi^\dagger(x) \left[\frac{\hbar^2}{2m} \nabla^2 + V_{\text{DQD}}(x) + V_{\text{PHOT}}(x) + V_{\text{PHON}}(x) \right] \Psi(x), \quad (2)$$

where

$$\Psi^\dagger(x) = e_L^\dagger \phi_L(x) + e_R^\dagger \phi_R(x). \quad (3)$$

Here, $V_{\text{DQD}}(x)$ is an electrostatic potential describing the DQD and $\Psi(x)$ is the field operator for DQD electrons. A realistic $V_{\text{DQD}}(x)$ can be obtained from a Hartree-Fock modeling of the DQD. Here we assume that only two energy levels of the DQD are relevant and use the variational basis $\phi_{L,R}(x) \sim$

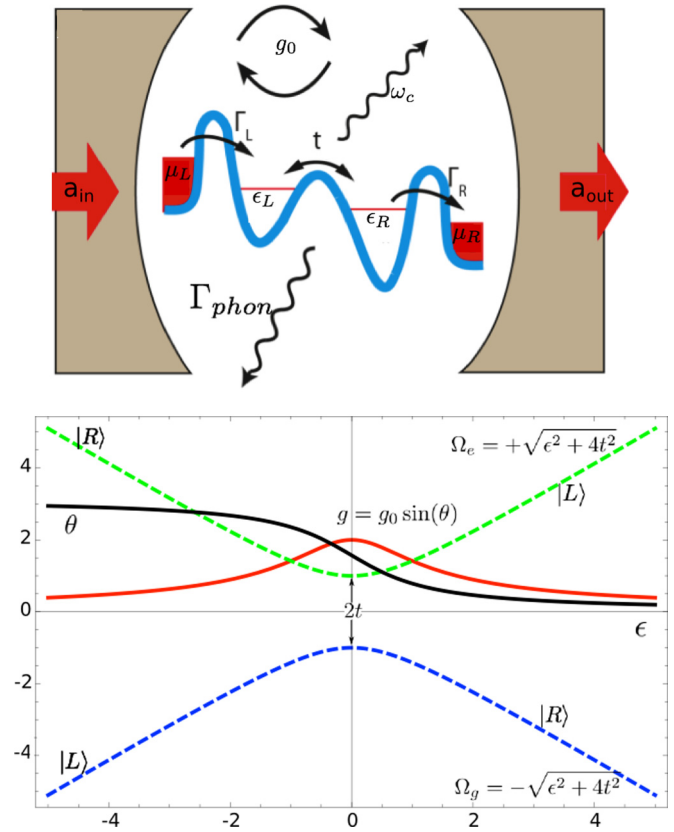


FIG. 1. (Color online) (Upper panel) Schematic of the DQD cQED experiment (adapted from Ref. [4]). The DQD is driven out of equilibrium by the application of a finite source-drain bias, which forces electrons to tunnel through the DQD. Due to electric dipole coupling between the trapped charge in the DQD and the electric field of the cavity, tunneling events in the DQD can influence the electromagnetic field inside the microwave cavity and vice versa. (Lower panel) Energetics of the DQD and effective light-matter coupling g (red) are shown as a function of detuning ϵ (x axis). Mixing angle θ as a function of detuning ϵ (x axis) is seen in black (solid).

$e^{-\frac{(x \pm d/2)^2}{2a^2}}$, where d is the interdot distance and a denotes the dot size. Using orthogonality properties of the basis functions $\phi_L(x), \phi_R(x)$, we get

$$H_{\text{DQD}} = \epsilon_L e_L^\dagger e_L + \epsilon_R e_R^\dagger e_R + t(e_L^\dagger e_R + e_R^\dagger e_L), \quad (4)$$

where t is the tunnel coupling between the two dots given by

$$t \equiv \int_{-\infty}^{+\infty} dx V_{\text{DQD}}(x) \phi_L(x) \phi_R(x) \quad (5)$$

and $\epsilon_i \equiv \int_{-\infty}^{+\infty} dx V_{\text{DQD}}(x) |\phi_i(x)|^2$ are the bare energy levels of the two QDs. The effective potential created on the quantum dot due to the cavity field is given by

$$V_{\text{photon}}^{L,R}(x) = \alpha_{L,R}(a + a^\dagger). \quad (6)$$

Plugging Eq. (6) into Eq. (2) gives the light-matter coupling within the constant interaction model

$$H_{\text{DQD-R}} = g_0 \tau_z (a + a^\dagger), \quad (7)$$

TABLE I. Parameter values from Ref. [4].

Cavity and driving frequency ω_c	32.5 μeV	7.9 GHz
Cavity loss rate κ	0.0082 μeV	2.0 MHz
Light-matter coupling g_0	0.0662 μeV	16 MHz
Elastic tunneling t	16.4 μeV	4.0 GHz
Drain tunneling rate Γ_R	16.56 μeV	4.0 GHz
Source tunneling rate Γ_L	16.56 μeV	4.0 GHz
Phonon cutoff frequency ω_{cutoff}	256 μeV	62 GHz
ω_D	60 μeV	14.5 GHz
Scaling frequency ω_0	32.8 μeV	7.9 GHz
j_{piezo} (unknown)	5.96 μeV	1.44 GHz
j_{def} (unknown)	0.0 μeV	0 GHz
Speed of sound in SiN c_s	11000 m/s	
Size of an individual quantum dot a	50 nm	
Temperature T	8 mK	
Gaussian noise σ	25 μeV	6.1 GHz

where g_0 is the coupling strength and $\tau_z \equiv |L\rangle\langle L| - |R\rangle\langle R|$. The value of g_0 is taken (Table I) to be close to the experimental setup and falls under the paradigm of weak light-matter coupling.

We next study the effective potential created by phonons. There are two important phonon contributions [21]: those of the substrate (deformation potential and piezoelectric) and the nanowire. This results in a total phonon spectral function

$$J(\omega) = J_{B,\text{piezo}}(\omega) + J_{B,dp}(\omega) + J_{NW}(\omega), \quad (8)$$

where we have

$$J_{B,dp}(\omega) = j_{dp} \left(\frac{\omega}{\omega_0} \right)^3 e^{-\frac{\omega^2}{\omega_D^2}} \left[1 - \sin\left(\frac{\omega}{\omega_{DD}} \right) \right], \quad (9)$$

$$J_{B,\text{piezo}}(\omega) = j_{\text{piezo}} \left(\frac{\omega}{\omega_0} \right) e^{-\frac{\omega^2}{\omega_D^2}} \left[1 - \sin\left(\frac{\omega}{\omega_{DD}} \right) \right], \quad (10)$$

and $J_{NW}(\omega)$ can be obtained from modeling with experimentally extracted parameters [21]. In Eq. (9) and Eq. (10), $\omega_D = \frac{\sqrt{2}c_s}{a_z}$ and $\omega_{DD} = \frac{c_s}{d}$ and ω_0 is a scaling parameter introduced for convenience which we choose to be 32.8 μeV . Here c_s is the speed of phonon propagation ($c_s \sim 11\,000$ m/s) and d is the distance between quantum dots ($d \sim 120$ nm for our structure), each with an axial confinement (dot size) given by $a_z \sim 40$ –50 nm and the radius of the nanowire is estimated to be around $r \sim 50$ nm.

The coupling strength constants j_{piezo} and j_{dp} are hard to independently estimate with the available experimental data [4,21]. However, for the purpose of this paper we choose $j_{\text{piezo}} \approx 5.96$ μeV and $j_{dp} \approx 0$ to fit the available experimental data. We do not find significant change in the features if both are taken to be nonzero except in the asymptotic behavior which we elaborate on in later sections.

After incorporating the effect of cavity photons, phonons, and the DQD potential, one is left with having to treat the coupling with electron leads at finite source-drain bias. For the treatment of the lead electrons, the reader is referred to Appendix A.

After integrating out the lead electrons and phonons we arrive at the following master equation for the DQD-cavity system with a coherent microwave drive of frequency ω_d

incident from the left waveguide:

$$\begin{aligned} \dot{\rho} = & -i[H, \rho] + \gamma_{\downarrow} \mathcal{D}(\sigma_-) + \gamma_{\uparrow} \mathcal{D}(\sigma_+) + \gamma_{\phi} \mathcal{D}(\sigma_z) + \kappa \mathcal{D}(a) \\ & + \Gamma_L \cos^2\left(\frac{\theta}{2}\right) \mathcal{D}(|e\rangle\langle 0|) + \Gamma_L \sin^2\left(\frac{\theta}{2}\right) \mathcal{D}(|g\rangle\langle 0|) \\ & + \Gamma_R \sin^2\left(\frac{\theta}{2}\right) \mathcal{D}(|0\rangle\langle e|) + \Gamma_R \cos^2\left(\frac{\theta}{2}\right) \mathcal{D}(|0\rangle\langle g|), \\ H = & \frac{\Omega}{2} \sigma_z + \omega_c a^{\dagger} a + g(\sigma_+ a + \sigma_- a^{\dagger}) \\ & + i\sqrt{\frac{\kappa}{2}} E \cos(\omega_d t)(a^{\dagger} - a), \end{aligned} \quad (11)$$

where Γ_L and Γ_R are the effective dot-lead tunneling rates which encode the information of the voltage bias (see Appendix A). The system Hamiltonian (11) takes the form of a coherently driven Jaynes-Cummings model for a two level system with transition frequency $\Omega = \sqrt{\epsilon^2 + 4t^2}$. Here $\sigma_{z,\pm}$ are pseudospin operators for the diagonal DQD levels e and g ,

$$|e\rangle = \cos\left(\frac{\theta}{2}\right)|L\rangle + \sin\left(\frac{\theta}{2}\right)|R\rangle, \quad (12)$$

$$|g\rangle = -\sin\left(\frac{\theta}{2}\right)|L\rangle + \cos\left(\frac{\theta}{2}\right)|R\rangle, \quad (13)$$

with $\theta = \arctan(2t/\epsilon)$ and $g = g_0 \sin(\theta)$ is the ϵ -dependent effective vacuum Rabi frequency (see Fig. 1). The last term in the Hamiltonian Eq. (11) is the coherent microwave tone of amplitude E incident from the left waveguide (we assume the coupling to the left and right waveguides are symmetric, $\kappa_L = \kappa_R = \kappa/2$).

Turning to the dissipative terms in (11) described by the Lindblad dissipators $\mathcal{D}(C) = C\rho C^{\dagger} - [C^{\dagger}C, \rho]/2$, $\kappa = \kappa_L + \kappa_R$ is the total cavity loss rate to the left and right waveguides and $\Gamma_{L,R}$ are the tunneling from the source reservoir (L) into the left dot and from the right dot into the drain reservoir (denoted by R). We consider a situation (see Fig. 1) where $V_{SD} = \mu_L - \mu_R > 0$ and that for any left-right detuning $\epsilon \equiv \epsilon_L - \epsilon_R$ the levels are always far from the respective Fermi levels. Here, we assume that we operate within the finite bias triangle corresponding to a transport cycle with one additional electron sequentially tunneling through the DQD from left to right. While $|L\rangle$ and $|R\rangle$ denote single particle levels, we use it here in this regime also to denote the many-body energy levels with one additional electron on the left dot and the right dot, respectively. Additionally $|0\rangle$ denotes the “empty” dot, i.e., the dot in its original configuration before the injection of an electron. Finally, $\gamma_{\downarrow}$, $\gamma_{\uparrow}$, γ_{ϕ} are the phonon relaxation, pumping, and pure dephasing contributions that follow from Eq. (8) given by

$$\gamma_{\uparrow}(\epsilon) = \sin^2(\theta) \bar{n}(\Omega, T) J(\Omega), \quad (14)$$

$$\gamma_{\downarrow}(\epsilon) = \sin^2(\theta) [1 + \bar{n}(\Omega, T)] J(\Omega), \quad (15)$$

$$\gamma_{\phi}(\epsilon) = \cos^2(\theta) [1 + 2\bar{n}(0, T)] J(0). \quad (16)$$

We note that these rates are strong functions of the left-right detuning ϵ which will have important ramifications for the results obtained. Here $\bar{n}(\omega, T) = \frac{1}{e^{\omega/T} - 1}$ denotes the mean phonon occupation at frequency ω and temperature T . We

assume that T is finite but small with respect to all energy scales of the Hamiltonian. Because $J \sim \omega^s$ with $s > 0$, the pure dephasing term γ_ϕ can be neglected (see also Refs. [24,25]). In what follows, all numerical data presented and compared to the analytic and experimental quantities are computed through the exact numerical solution (with a cavity photon number cutoff) of Eq. (11).

III. ANALOGY WITH SINGLE-ATOM LASER AND BEYOND

The master equation derived in the previous section is not convenient to gain an analytic insight into the physics of the DQD-cavity amplifier. Progress can be made by integrating out the empty dot state and setting up an analogy to the standard theory of the single-atom laser/amplifier [26],

$$\begin{aligned} \dot{\rho} = & -i[H, \rho] + \Gamma_\downarrow \mathcal{D}(\sigma_-) + \Gamma_\uparrow \mathcal{D}(\sigma_+) \\ & + \Gamma_\phi \mathcal{D}(\sigma_z) + \kappa \mathcal{D}(a), \end{aligned} \quad (17)$$

with effective pumping, relaxation, and pure dephasing rates given by¹

$$\Gamma_\uparrow = \gamma(1 + W_p)/2, \quad (18)$$

$$\Gamma_\downarrow = \gamma(1 - W_p)/2, \quad (19)$$

$$\Gamma_\phi = (\gamma_d - \gamma/2)/2, \quad (20)$$

where (also see Fig. 2)

$$\begin{aligned} \gamma = & \gamma_\uparrow + \gamma_\downarrow + \Gamma_R \left[\cos^4\left(\frac{\theta}{2}\right) + \sin^4\left(\frac{\theta}{2}\right) \right] \\ & - \frac{\cos \theta}{1 + \frac{\Gamma_R}{2\Gamma_L}} \frac{\Gamma_R}{2\Gamma_L} [\gamma_\uparrow - \gamma_\downarrow + \Gamma_R \cos \theta], \end{aligned} \quad (21)$$

$$W_p \equiv \frac{1}{\gamma} \frac{\gamma_\uparrow - \gamma_\downarrow + \Gamma_R \cos \theta}{1 + \Gamma_R/2\Gamma_L}, \quad (22)$$

$$\gamma_d = 2\gamma_\phi + (\gamma_\uparrow + \gamma_\downarrow + \Gamma_R)/2. \quad (23)$$

This master equation describes a single-atom laser with active levels e and g and effective pump and dissipation terms that are complicated functions of the interdot detuning ϵ , through the dependences of the mixing angle $\theta(\epsilon)$ and the phonon dissipation rates $\gamma_\downarrow(\epsilon)$, $\gamma_\uparrow(\epsilon)$, and $\gamma_\phi(\epsilon)$. We note that the steady-state solution ρ_{ss} of this master equation contains full information about the photonic as well as the electronic transport observables. In this section we focus on the photonic aspects and, in particular, the transmission, phase response, photon number, emission spectrum, and the photon statistics. The reader is referred to Appendix B for the details of calculations leading to the transmission and the phase response.

Turning the drive off ($E = 0$) we obtain the following equations of motion in the rotating frame with respect to ω_l , where ω_l is an emergent frequency scale to be determined

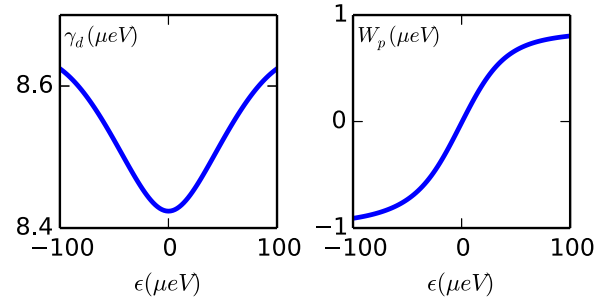


FIG. 2. (Color online) (Left panel) γ_d [Eq. (24)] and (right panel) W_p [Eq. (23)] as a function of detuning ϵ .

below:

$$\langle \dot{a} \rangle = -i\Delta_{cl}\langle a \rangle - \frac{\kappa}{2}\langle a \rangle - ig\langle \sigma_- \rangle, \quad (24)$$

$$\langle \dot{\sigma}_- \rangle = -(i\Delta_{ql} + \gamma_d)\langle \sigma_- \rangle + ig\langle a\sigma_z \rangle, \quad (25)$$

$$\langle \dot{\sigma}_z \rangle = \gamma(W_p - \langle \sigma_z \rangle) + 2ig(\langle a^\dagger \sigma_- \rangle - \langle a\sigma_+ \rangle). \quad (26)$$

Here, $\Delta_{cl} = \omega_c - \omega_l$, $\Delta_{ql} = \Omega - \omega_l$, $W_p(\epsilon)$ is the effective incoherent pump power measured in terms of the unsaturated inversion it creates, $\gamma_d(\epsilon)$ is the dipole dissipation rate, and $\gamma(\epsilon)$ is the population relaxation rate [see Eqs. (21)–(23)]. We show the strong ϵ dependence of the pump rate W_p and the dipole dissipation rate γ_d in Fig. 2.

These equations can be solved in the steady state for $\langle a \rangle_{ss}$, $\langle \sigma_- \rangle_{ss}$, and $\langle \sigma_z \rangle_{ss}$ by the semiclassical factorization $\langle aO \rangle \approx \langle a \rangle \langle O \rangle$ yielding equations that are equivalent to Maxwell-Bloch equations. In this approximation Eqs. (24) to (26) reduce to

$$\langle a \rangle_{ss} \left(1 - \frac{1}{\frac{\kappa}{2} + i\Delta_{cl}} \frac{g^2}{i\Delta_{ql} + \gamma_d} \frac{W_p}{1 + G|\langle a \rangle_{ss}|^2} \right) = 0,$$

which is an equation for $|\langle a \rangle_{ss}|$ and ω_l leaving the phase of the field undetermined. Here $G = \frac{4g^2\gamma_d}{\gamma(\gamma_d^2 + \Delta_{ql}^2)}$ and the product GW_{th} is the slope efficiency.

These can be solved and yield the solution

$$\begin{aligned} \langle a^\dagger a \rangle_{ss} = |\langle a \rangle_{ss}|^2 = 0, \quad W_p < W_{th}, \\ = \frac{1}{G} \left(\frac{W_p}{W_{th}} - 1 \right), \quad W_p > W_{th}. \end{aligned} \quad (27)$$

Hence this semiclassical solution predicts lasing above the threshold pump power $W_p = W_{th} = \frac{\kappa(\gamma_d^2 + \Delta_{ql}^2)}{2g^2\gamma_d}$. It is to be noted that, although the semiclassical solution predicts a sharp threshold, this turn on is rather smooth when fluctuations are taken into account.

The same equation provides the line-pulling formula for the lasing frequency

$$\omega_l = \omega_c \frac{\gamma_d}{\gamma_d + \kappa/2} + \Omega \frac{\kappa/2}{\kappa/2 + \gamma_d}. \quad (28)$$

The steady-state inversion is clamped to its threshold value above the threshold

$$\langle \sigma_z \rangle_{ss} = W_p, \quad W_p < W_{th} = W_{th}, \quad W_p > W_{th}. \quad (29)$$

It's important to note that the semiclassical solutions have a limited validity for a single-atom laser. This stems from the fact

¹These effective rates are plotted in Fig. 10 (Appendix A) as a function of detuning.

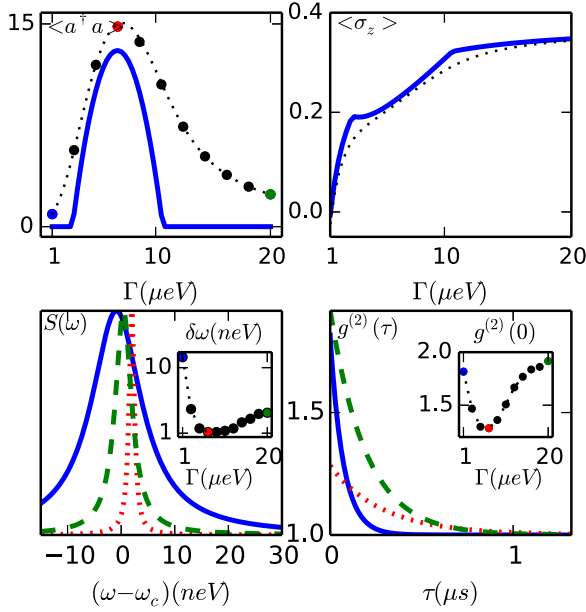


FIG. 3. (Color online) (Top left) Photon number as a function of dot-lead tunneling rate. The solid line (blue) is semiclassical treatment and the dots are exact numerical results. (Top right) Inversion as a function of dot-lead tunneling rate where the dots are numerically exact results and the solid line (blue) is semiclassical treatment. (Bottom left) Plot showing power spectrum $S(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \langle a^\dagger(t)a(0) \rangle dt$ with varying dot-lead tunneling rate. The different colors (blue, green, red) or solid, dashed, and dotted lines, respectively, represent different dot-lead tunneling. Inset represents the full width at half maximum of the power spectrum as a function of dot-lead tunneling. (Bottom right) Plot of $g^{(2)}(\tau) = \frac{\langle a^\dagger a^\dagger(\tau)a(\tau)a \rangle}{\langle a^\dagger a \rangle^2}$ for different dot-lead tunneling rates. The inset represents $g^{(2)}(0)$ as a function of dot-lead tunneling rate. We consider the symmetric case $\Gamma_L = \Gamma_R = \Gamma$. The values are the same as in Table I except where mentioned. Blue (solid), red (dotted), and green (dashed) represent $\Gamma = 1, 6.7, 20 \mu\text{eV}$, respectively.

that the gain medium is not a collective system described by a large spin but a single spin for which quantum fluctuations cannot be ignored. To illustrate this we plot in Fig. 3 the semiclassical photon number versus the exact photon number obtained by the numerical solution of the master equation [Eq. (11)] as well as the inversion. Here we ramp up the pump power by biasing the gates between the left dot and the source reservoir (i.e., Γ_L).

We note the existence of a second threshold above which lasing turns off. This stems from the quantum nature of the gain medium. The same bath that via a complex interplay of phononic and fermionic baths gives rise to the pumping term (W_p) also gives rise to the quenching of the DQD dipole (via γ_d). While W_p saturates (as, e.g., Γ_L is ramped up) γ_d monotonically increases. Above some critical tunneling rate lasing is quenched, signaled by a sharp semiclassical second threshold visible in Fig. 3. It's important to note that the laser threshold can only be reached in the strong coupling regime $g^2(\epsilon) > \kappa\gamma_d(\epsilon)$.

A further indication that what we have here is not a conventional laser can be seen from the emission spectrum and in particular the photon statistics. In Fig. 3 we show the emission spectrum $S(\omega)$ and the second order correlation

function $g^{(2)}(\tau)$ of photons emitted from the cavity:

$$S(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \langle a^\dagger(t)a(0) \rangle dt \quad (30)$$

and

$$g^{(2)}(\tau) = \frac{\langle a^\dagger a^\dagger(\tau)a(\tau)a \rangle}{\langle a^\dagger a \rangle^2}. \quad (31)$$

We note that the broad emission spectrum of the DQD-cavity system below the semiclassical threshold narrows by an order of magnitude as we approach maximum cavity emission, and then broadens again. The behavior above the threshold is consistent with the Shallow-Townes linewidth $\delta\omega \propto 1/\langle a^\dagger a \rangle_{ss}$. We also note that there is a strong frequency renormalization of the spectral peak from the cavity frequency below the threshold to the laser frequency given by the line-pulling formula Eq. (29). Turning to the photon statistics, we see in the right panel that below the threshold we have effectively a thermalized photon gas in the cavity [$g^{(2)}(0) \sim 2$], while above the threshold the distribution $g^{(2)}(0)$ moves away from its thermal value but remains always above 1. For larger Γ , the distribution asymptotically approaches 2 again.

We now consider the transmission $A = \frac{\langle a \rangle \sqrt{2\kappa}}{E}$ by turning on E while remaining in the linear response regime. This can be obtained by the nonlinear steady-state susceptibility of the amplifier (see Appendix B)

$$A(\epsilon) = \frac{i\kappa/2}{i\kappa/2 + \Sigma(\epsilon)}, \quad (32)$$

$$\text{Im}[\Sigma] = \frac{-g^2 W_p \gamma_d}{\Delta_{qd}^2 + \gamma_d^2}, \quad (33)$$

$$\text{Re}[\Sigma] = -\frac{g^2 W_p \Delta_{qd}}{\Delta_{qd}^2 + \gamma_d^2}. \quad (34)$$

In what follows, instead of analyzing the ω_d dependence of the transmission amplitude, we will fix the incident microwave signal to the cavity frequency $\omega_d = \omega_c$ and analyze the dependence on the two-level system transition frequency via the tuning of ϵ . The self-energy correction emerges due to the coupling of the DQD electron to phonon and electron baths. In Fig. 4 we show the transmission and phase response as a function of detuning. In particular, we show the trend as one increases the lead-dot tunneling rate (Γ'), light-matter coupling (g'), and the interdot tunneling (t'). As reference (g, t, Γ) we take the experimental values for data analyzed in Sec. V.

The most distinctive feature of the transmission amplitude is the gain peak on the positive side of detuning $\epsilon > 0$. There is a significant renormalization of the position of the gain peak with respect to a bare current-driven DQD which would be at resonance, $\omega_d = \Omega$ or $\epsilon_{\text{opt}} = \sqrt{\omega_d^2 - 4t^2}$. Assuming the relaxation rate arising due to phonons is small compared to the dot-lead tunneling rate (as is the case for data presented in Sec. V), we arrive at the following expression for the optimal ϵ_{opt} at which maximal gain results:

$$\epsilon_{\text{opt}} \sim \frac{2\sqrt{2}t\sqrt{-16t\omega_c + 4\omega_c^2 + \Gamma^2 + 16t^2}}{\sqrt{-224t\omega_c + 44\omega_c^2 + 11\Gamma^2 + 272t^2}}. \quad (35)$$

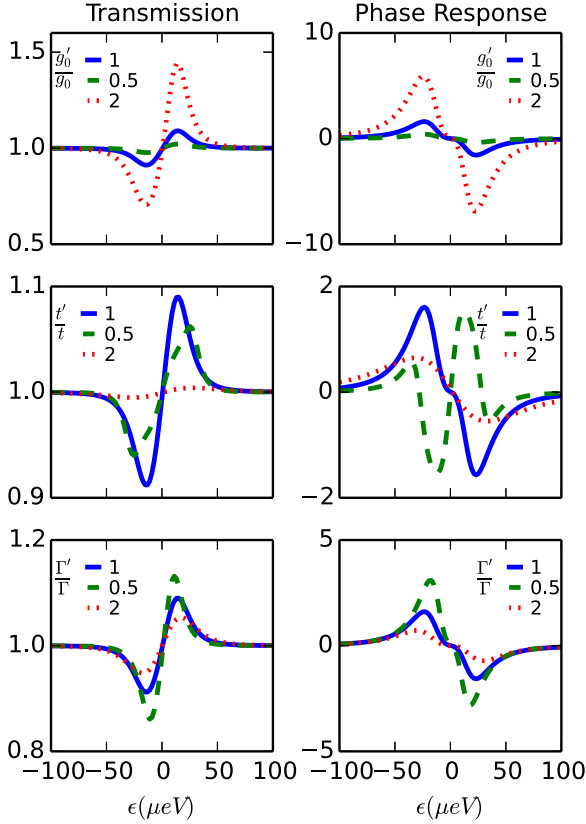


FIG. 4. (Color online) Plot showing analytical results for transmission (left column) and phase response (right column), i.e., argument in degrees of Eq. (32) versus detuning. From top to bottom panel is the behavior as one changes (i) light-matter coupling g_0 , (ii) interdot tunneling t , and (iii) dot-lead tunneling rates Γ_L, Γ_R . The analytical plot is generated using Eq. (32).

$$I^{\text{PHON}} = \frac{\Gamma_L \Gamma_R t^2}{\Gamma_L \epsilon^2 + t^2 (2\Gamma_L + \Gamma_R)} + \gamma_{\downarrow} \frac{\Gamma_L^2 \epsilon [2t^2 (\sqrt{4t^2 + \epsilon^2} + 2\epsilon) + \epsilon^2 (\sqrt{4t^2 + \epsilon^2} + \epsilon)]}{2[\Gamma_L \epsilon^2 + t^2 (2\Gamma_L + \Gamma_R)]^2} + O(\gamma_{\downarrow}^2). \quad (38)$$

One important consequence of including the phonon contribution is the induced asymmetry [27] in the current-detuning dependence as opposed to a symmetric Lorentzian obtained [Eq. (37)] when phonons are neglected.

In Fig. 5, we show the phonon- and photon-mediated current contributions to the total nonequilibrium steady-state current as a function of detuning. Close to the peak current on the positive detuning side, the photon-mediated component gives rise to an additional current due to the stimulated emission being maximal in that range. This additional photon-induced current on the positive detuning side (see right panel of Fig. 5) occurs in a detuning window where the system is above the lasing threshold,

$$\epsilon : W_P(\epsilon) > W_{th}(\epsilon). \quad (39)$$

It is worthwhile to analyze the asymptotic behavior of the dc current Eq. (36). For large detuning ϵ the threshold is pushed further such that $W_P(\epsilon) < W_{th}(\epsilon)$. This means that one is left only with the coherent tunneling current and the phonon-induced current.

We note that this optimal value is the result of the saturation of the effective pump rate (Fig. 2) and the quenching of the light-matter interaction strength g with increasing ϵ (Fig. 1). The above expression is valid only when the observed maximal gain is at relatively small detuning. For the realistic parameters of the experiment, the above formula for optimal detuning seems to be the point where the peak is theoretically predicted.

IV. NONEQUILIBRIUM ELECTRON CURRENT

In this section we analyze the nonequilibrium steady-state current below and above the laser threshold. In the presence of nonzero phonon and photon coupling, a rather compact analytic expression can be obtained in the small Γ/t limit

$$\begin{aligned} I_t^{\Gamma < 1} &= \frac{\Gamma_R}{2(1 + \Gamma_R/2\Gamma_L)} (1 - \langle \sigma_z \rangle_{ss} \cos \theta) \\ &\quad + \Gamma_R \sin \theta \operatorname{Re}(\langle \sigma_- \rangle_{ss}) \\ &= \frac{\Gamma_R}{2(1 + \Gamma_R/2\Gamma_L)} (1 - \langle \sigma_z \rangle_{ss} \cos \theta). \end{aligned} \quad (36)$$

This expression is valid both below and above threshold. Here we can insert $\langle \sigma_z \rangle$ from Eq. (29) and $\langle \sigma_- \rangle = 0$. Equation (36) has a complex combination of phonon and photon contribution that can be isolated.

In the limit of very small phonon and photon coupling the current is mediated only by coherent interdot tunneling and is given as a simplified limit of Eq. (36) by

$$I^{\text{no phonon, no photon}} = \frac{\Gamma_L \Gamma_R t^2}{\Gamma_L \epsilon^2 + t^2 (2\Gamma_L + \Gamma_R)}. \quad (37)$$

Below the lasing threshold, we have only phonon contribution (in addition to the coherent interdot tunneling contribution) and this gives us the following:

The tail on the positive detuning side is dominated by phonon-induced processes. We note that, for $\epsilon \rightarrow \infty$, the current behaves as

$$I_{|\epsilon| \rightarrow \infty} = \frac{\Gamma_R t^2}{\epsilon^2} + \gamma_{\downarrow}(\epsilon). \quad (40)$$

Therefore, the large detuning tail of the current [Eq. (40)] is a competition between coherent tunneling (ϵ^{-2}) and the effect of phonons which could be of the deformation ($\epsilon e^{-\epsilon^2/\omega_D^2}$) or piezoelectric type ($\epsilon^{-1} e^{-\epsilon^2/\omega_D^2}$).

V. EXPERIMENTS AND COMPARISON TO THEORETICAL PREDICTIONS

Here, we will present experimental data and compare them to predictions of the theoretical models discussed in the previous sections. We also take into account charge noise [4] of magnitude $\sigma \approx 25 \mu\text{eV}$ by averaging any theoretical result $F(\epsilon)$ by $\bar{F}(\epsilon) = \int_{-\infty}^{\infty} d\tilde{\epsilon} e^{-\frac{(\epsilon-\tilde{\epsilon})^2}{2\sigma^2}} F(\tilde{\epsilon})$. The present experimental parameters are in a regime where there is no lasing. The

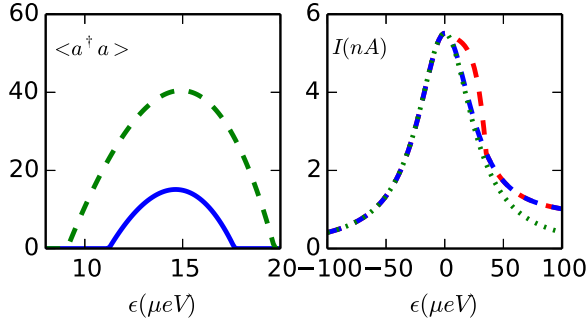


FIG. 5. (Color online) (Left) Plot showing photon number versus detuning with and without phonons. In particular, one clearly notices the phonon reduced photon emission. Green (dashed) is without phonons and blue (solid) is with phonons. We used $g'_0 = 5g_0$. (Right) Plot showing current (analytical) versus detuning. The three plots of nonequilibrium current here are (green, dotted) without phonons and photons, (blue, dashed without the bump) with only phonons, and (red, dashed with the bump) with both phonons and photons. We used a light-matter coupling $g'_0 = 10g_0$. The analytical plots are generated using Eqs. (27) and (36).

effective pump W_p in the experiments is much smaller than the threshold W_{th} (approximately $W_p/W_{th} < 0.024$).

In Fig. 6, we first analyze small current data where the dot-lead coupling can be treated perturbatively. As can be noted in both panels of Fig. 6, the agreement between theory and experiment is quite good. This suggests that treating phonons as a reservoir in the Born-Markov approximation is sufficient to capture the current data, where phonon coupling leads to a pronounced asymmetry in the dependence of the dc current on detuning. The good fit between theory and experiment here demonstrates the validity of treating both the electronic and the phonon reservoir in Born-Markov approximation.

We now turn to large current data recorded on another sample. As can be seen in Fig. 7, when the current is large, there is sizable discrepancy between theory and experiment. Since there is significant evidence that phonons can be treated via the Born-Markov approach, we can therefore identify this discrepancy to be attributed to the strong coupling of the dot to the leads which invalidates the perturbative treatment of the leads underlying our master equation.

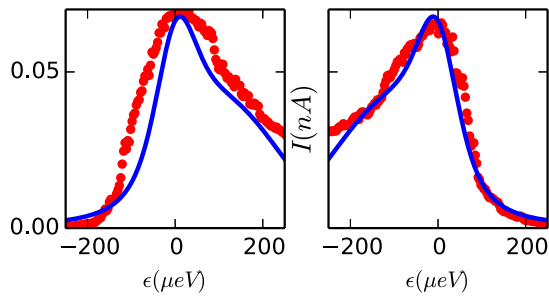


FIG. 6. (Color online) Comparison of our nonequilibrium current analytical expression [Eq. (36)] through a system with small DQD-leads coupling for negative (left) and positive (right) source drain bias with the experimental data. The best fit was for $t = 29 \mu\text{eV}$, $c'_{\text{piezo}} = 1.7 \mu\text{eV}$, $\Gamma_R = 0.77 \mu\text{eV}$, and $\Gamma_L = 1.8 \mu\text{eV}$. Theory is in blue (solid) and experiment is in red (dots). The analytical fits are generated using Eq. (36).

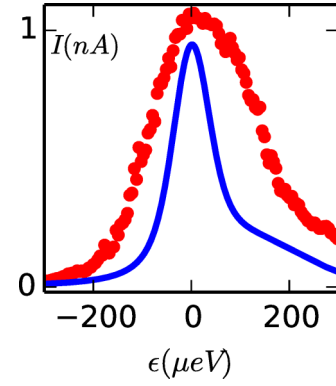


FIG. 7. (Color online) Theoretical (blue, solid) versus experimental (red, dots) fit of the nonequilibrium steady-state current. The analytical fits are generated using Eq. (36).

We next study photonic properties of the system at finite voltage bias. Figure 8 shows the transmission, phase response for a case when the bias is high. We see that there is a sizable discrepancy between the theoretical data and experiments. This is again attributed to the fact that the dot-lead coupling is too high for a Born-Markov approach to be valid (for the electron leads). We find that the transmission as measured in experiment is higher than the one predicted by theory. The gain on the positive detuning side seen in Fig. 9 is essentially arising from an interdot charge transition from a higher to a lower energy level.

VI. CONCLUSION

We provided a first-principles derivation of the master equation for a DQD coupled to a single cavity mode, considering systematically both the presence of phonons and the effect of leads at finite voltage bias. We derived an effective model of a rather unconventional single-atom laser/amplifier and investigated all relevant quantities (such as photon transmission, phase response, threshold, spectrum, photon number, and $g^{(2)}$ correlation function). Phonons and the leads under finite bias are established to be the source of relaxation, pumping, and pure dephasing mechanisms. We obtain the analytical results for the dc current, photon flux, photon transmission, and phase response below and above threshold and these results are

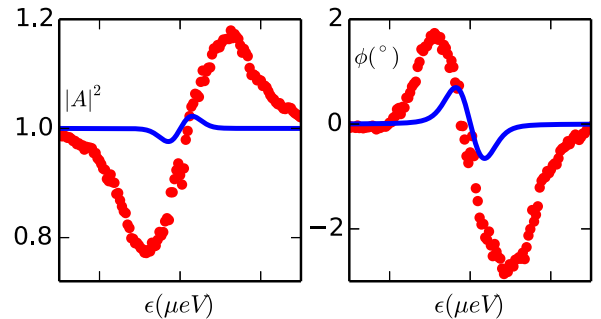


FIG. 8. (Color online) Theoretical (blue, solid) versus experimental (red, dots) fit for transmission (left) and phase response (right). The analytical fits are generated using Eq. (32).

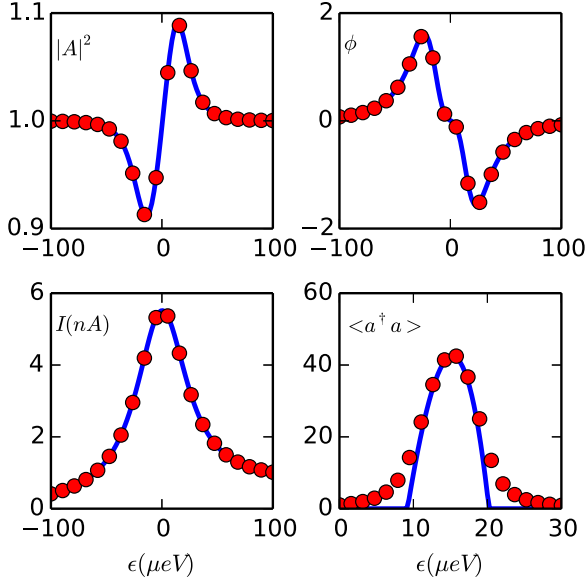


FIG. 9. (Color online) (Top left) Photon transmission versus detuning (ϵ). (Top right) Phase response versus detuning. (Bottom left) Nonequilibrium steady-state current versus detuning. (Bottom right) Cavity photon number versus detuning. Analytics is solid line (blue), while exact numerics is red circles. Parameter values are given in Table I except for photon emission (bottom right), where $g_0 \rightarrow 5.3g_0$ so that we can go above threshold. The discrepancy in photon number exhibited in the bottom right figure is exactly what we would expect from a semiclassical approximation. The analytical results are generated from Eq. (32) (transmission and phase response), Eq. (27) (photon number), and Eq. (36) (dc electron current).

benchmarked by exact master equation simulations. When applicable, we compared our analytical and numerical results to existing experimental data of photon transmission, phase response, and nonequilibrium steady-state electron current. We believe that these results, both analytical and numerical, are of paramount importance in establishing the optimal parameter choice for future experiments to reach the strong-coupling regime and lasing [28–31], as studied before in atomic and solid-state cavity QED [32–34].

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APPENDIX A: ELECTRON TRANSPORT

In this appendix we analyze transport in the special case when there is at most one electron on the DQD.

The starting Hamiltonian is

$$H = H_{\text{DQD}} + H_{\text{leads}} + H_{\text{leads-DQD}}, \quad (\text{A1})$$

$$H_{\text{DQD}} = \varepsilon_L c_L^\dagger c_L + \varepsilon_R c_R^\dagger c_R + t(c_L^\dagger c_R + c_R^\dagger c_L), \quad (\text{A2})$$

$$H_{\text{leads}} = \sum_{kj} \omega_{kj} b_{kj}^\dagger b_{kj}, \quad (\text{A3})$$

$$H_{\text{leads-DQD}} = \sum_{kj} (t_{kj} b_{kj} c_j^\dagger + \text{H.c.}), \quad (\text{A4})$$

where $j \in \{L, R\}$, $c_L^\dagger \equiv |L\rangle \langle 0|$, and $c_R^\dagger \equiv |R\rangle \langle 0|$ are the creation operators for an electron on the left and right QDs and $b_{kj}^\dagger = |k, j\rangle \langle 0|$ is the creation operator for an electron in state k in the j th reservoir.

In the Schrödinger picture the Born-Markov second order master equation takes the form [35]

$$\begin{aligned} \dot{\rho}_{\text{DQD}}(t) = & -i[H_{\text{DQD}}, \rho_{\text{DQD}}(t)] - \int_0^\infty d\tau [H_{\text{leads-DQD}}, \\ & \times [H_{\text{leads-DQD}}(\tau), \rho_{\text{DQD}}(t) \otimes \rho_{\text{leads}}]] \rho_{\text{leads}}, \end{aligned} \quad (\text{A5})$$

where $O(\tau) \equiv e^{i(H_{\text{DQD}} + H_{\text{leads}})\tau} O e^{-i(H_{\text{DQD}} + H_{\text{leads}})\tau}$ is the interaction picture operator O at time τ .

Therefore, in order to find the master equation we need to know the time dependence of $H_{\text{leads-DQD}}$ in the interaction picture. For this we need to know the time dependence of $c_j^\dagger$ and $b_{kj}^\dagger$:

$$\begin{aligned} c_L^\dagger(t) &= \cos\left(\frac{\theta}{2}\right) e^{iE_e t} c_e^\dagger - \sin\left(\frac{\theta}{2}\right) e^{iE_g t} c_g^\dagger, \\ c_R^\dagger(t) &= \sin\left(\frac{\theta}{2}\right) e^{iE_e t} c_e^\dagger + \cos\left(\frac{\theta}{2}\right) e^{iE_g t} c_g^\dagger, \\ b_{kj}^\dagger(t) &= e^{i\varepsilon_{kj} t} b_{kj}^\dagger, \end{aligned} \quad (\text{A6})$$

where $c_e^\dagger = |e\rangle \langle 0|$, $c_g^\dagger = |g\rangle \langle 0|$ are the creation operators for electrons on the DQD in the excited and the ground state and $E_{e,g,0}$, the eigenenergies of the DQD, are

$$E_e = \frac{\varepsilon_L + \varepsilon_R}{2} + \frac{\Omega}{2}, \quad (\text{A7})$$

$$E_g = \frac{\varepsilon_L + \varepsilon_R}{2} - \frac{\Omega}{2}, \quad (\text{A8})$$

$$E_0 = 0, \quad (\text{A9})$$

where $\Omega = \sqrt{(\varepsilon_L - \varepsilon_R)^2 + 4t^2}$.

To evaluate (A5) we use the fact that the leads are fermionic reservoirs at zero temperature with the chemical potential μ_j , and ignore the usually trivial effects of Lamb-shift type of terms. We need to evaluate the following types of integrals:

$$\begin{aligned} \left\langle \sum_k b_{kj} \right\rangle_{H_{\text{leads}}} &= \left\langle \sum_k b_{kj}^\dagger \right\rangle_{H_{\text{leads}}} = 0, \\ \left\langle \int_0^\infty d\tau \sum_k t_{kj}^* b_{kj}^\dagger(0) \sum_q t_{qj} b_{qj}(\tau) e^{i\omega_0 \tau} \right\rangle_{H_{\text{leads}}} \\ &= \int_0^\infty d\tau \sum_{k,q} t_{kj}^* t_{qj} \langle b_{kj}^\dagger(0) b_{qj}(0) \rangle_{H_{\text{leads}}} e^{i(\omega_0 - \omega_{qj})\tau} \\ &= \int_0^\infty d\tau \sum_{k,q} t_{kj}^* t_{qj} \delta(k, q) \Theta(\mu_j - \omega_{qj}) e^{i(\omega_0 - \omega_{qj})\tau} \end{aligned}$$

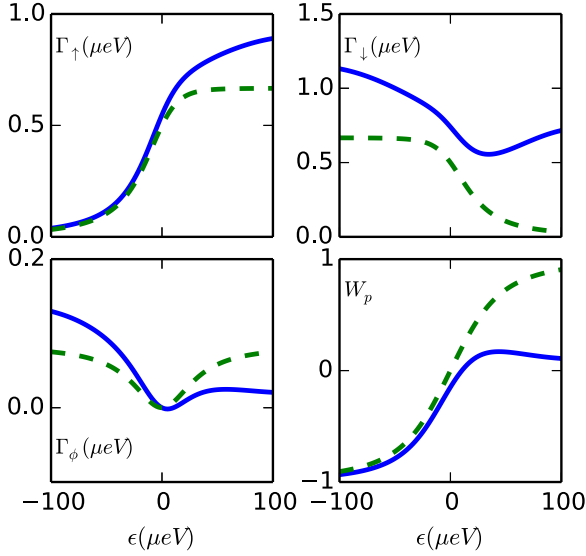


FIG. 10. (Color online) Top left: $\Gamma_{\uparrow}$; top right: $\Gamma_{\downarrow}$; bottom left: Γ_{ϕ} ; bottom right: W_p . $\Gamma_R = \Gamma_L = 1 \mu\text{eV}$; the rest of the experimental values are as in Table I. Green is without phonons; blue is with phonons.

$$\begin{aligned}
 &= \sum_k |t_{kj}|^2 \Theta(\mu_j - \omega_{kj}) \pi \delta(\omega_0 - \omega_{kj}) \\
 &= \frac{1}{2} J(\omega_0) \Theta(\mu_j - \omega_0),
 \end{aligned} \tag{A10}$$

where $J_j(\omega) \equiv 2\pi \sum_k |t_{kj}|^2 \delta(\omega - \omega_{kj})$ is the spectral density function of the j th reservoir and $\Theta(x)$ is the Heaviside step function. Above, ω_0 is just a variable and could be any frequency of interest (for example, E_e, E_g).

For simplicity, we evaluate the special case when $\mu_L > E_e, E_g > \mu_R$ and we assume that the reservoir spectral density function varies slowly between $\omega = E_e$ and $\omega = E_g$ such that we can make the approximation $J_j(E_g) \approx J_j(E_e)$. In this case we introduce (A6) into (A5) and use (A10) to obtain the following master equation:

$$\begin{aligned}
 \dot{\rho}_{\text{DQD}} = & -i[H_{\text{DQD}}, \rho_{\text{DQD}}] \\
 & + \Gamma_L \mathcal{D} \left[\cos\left(\frac{\theta}{2}\right) |e\rangle\langle 0| + \sin\left(\frac{\theta}{2}\right) |g\rangle\langle 0| \right] \\
 & + \Gamma_R \mathcal{D} \left[-\sin\left(\frac{\theta}{2}\right) |0\rangle\langle e| + \cos\left(\frac{\theta}{2}\right) |0\rangle\langle g| \right],
 \end{aligned} \tag{A11}$$

where $\Gamma_L = 1/2 J_L(E_e)$ and $\Gamma_R = 1/2 J_R(E_e)$.

We “secularize” Eq. (A11) as described in Refs. [36–38]. In this approximation, the excited and the ground state of the DQD are assumed to interact independently with the lead electrons. Secularization is valid as long as the linewidths of the ground and excited states of the DQD are smaller than the energy difference between them and the two states can be distinguished clearly. When this is not the case (i.e., $\Gamma_{L,R} \geq E_e - E_g$) we expect this approximation to break down. The

secularized master equation is

$$\begin{aligned}
 \dot{\rho}_{\text{DQD}} = & -i[H_{\text{DQD}}, \rho_{\text{DQD}}] + \Gamma_L \cos^2\left(\frac{\theta}{2}\right) \mathcal{D}(|e\rangle\langle 0|) \\
 & + \Gamma_L \sin^2\left(\frac{\theta}{2}\right) \mathcal{D}(|g\rangle\langle 0|) \\
 & + \Gamma_R \sin^2\left(\frac{\theta}{2}\right) \mathcal{D}(|0\rangle\langle e|) \\
 & + \Gamma_R \cos^2\left(\frac{\theta}{2}\right) \mathcal{D}(|0\rangle\langle g|),
 \end{aligned} \tag{A12}$$

$$H_{\text{DQD}} = \varepsilon_L e_L^\dagger e_L + \varepsilon_R e_R^\dagger e_R + t(e_L^\dagger e_R + e_R^\dagger e_L).$$

In the eigenbasis H_{DQD} has the form

$$H_{\text{DQD}} = \frac{\varepsilon_L + \varepsilon_R}{2} I_2 + \frac{\Omega}{2} \sigma_z = -\frac{\varepsilon_L + \varepsilon_R}{2} |0\rangle\langle 0| + \frac{\Omega}{2} \sigma_z, \tag{A13}$$

where we used the Pauli spin operators in the eigenbasis $\{|e\rangle, |g\rangle\}$ (i.e., $I_2 = |e\rangle\langle e| + |g\rangle\langle g|$, $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_+ = |e\rangle\langle g|$, and $\sigma_- = |g\rangle\langle e|$).

Since the zero state energy does not enter the dynamics anymore (this energy only mattered in obtaining the rates $\Gamma_{L,R}$), we can ignore the term proportional $|0\rangle\langle 0|$ with impunity to obtain the final Hamiltonian:

$$H_{\text{DQD}} = \frac{\Omega}{2} \sigma_z. \tag{A14}$$

APPENDIX B: EQUATIONS OF MOTION

1. Starting equations of motion

We start from the secularized master equation:

$$\begin{aligned}
 \dot{\rho} = & -i[H, \rho] + \gamma_{\downarrow} \mathcal{D}(\sigma_-) + \gamma_{\uparrow} \mathcal{D}(\sigma_+) + \gamma_{\phi} \mathcal{D}(\sigma_z) + \kappa \mathcal{D}(a) \\
 & + \Gamma_L \cos^2\left(\frac{\theta}{2}\right) \mathcal{D}(|e\rangle\langle 0|) + \Gamma_L \sin^2\left(\frac{\theta}{2}\right) \mathcal{D}(|g\rangle\langle 0|) \\
 & + \Gamma_R \sin^2\left(\frac{\theta}{2}\right) \mathcal{D}(|0\rangle\langle e|) + \Gamma_R \cos^2\left(\frac{\theta}{2}\right) \mathcal{D}(|0\rangle\langle g|), \\
 H = & \frac{\Omega}{2} \sigma_z + g(\sigma_+ a + \sigma_- a^\dagger) \\
 & + \omega_c a^\dagger a + i \sqrt{\frac{\kappa}{2}} E \cos(\omega_d t) (a^\dagger - a).
 \end{aligned} \tag{B1}$$

To solve the above problem we use linear response theory. Since E is a small perturbation we ignore the time dependent term $i \sqrt{\kappa/2} E \cos(\omega_d t) (a^\dagger - a)$ at first. We assume that the system will reach a steady state in which the photon electric field and DQD electric dipole oscillate at a frequency ω_l which we call the laser frequency and will be determined subsequently. Therefore, we look at the equations of motion in a frame rotating at ω_l :

$$\langle \dot{a} \rangle = -i \Delta_{cl} \langle a \rangle - \frac{\kappa}{2} \langle a \rangle - i g \langle \sigma_- \rangle, \tag{B2}$$

$$\begin{aligned}
 \langle \dot{\sigma}_- \rangle = & - \left(i \Delta_{ql} + 2\gamma_{\phi} + \frac{\gamma_{\uparrow} + \gamma_{\downarrow} + \Gamma_R}{2} \right) \langle \sigma_- \rangle \\
 & + i g \langle a \sigma_z \rangle,
 \end{aligned} \tag{B3}$$

$$\begin{aligned} \langle \dot{\sigma}_z \rangle = & - \left[2\gamma_\downarrow + \Gamma_R \sin^2 \left(\frac{\theta}{2} \right) \right] \frac{1}{2} (\langle I_2 \rangle + \langle \sigma_z \rangle) \\ & + \left[2\gamma_\uparrow + \Gamma_R \cos^2 \left(\frac{\theta}{2} \right) \right] \frac{1}{2} (\langle I_2 \rangle - \langle \sigma_z \rangle) \\ & + 2ig(\langle a^\dagger \sigma_- \rangle - \langle a \sigma_+ \rangle) + \Gamma_L \cos \theta \langle \sigma_0 \rangle, \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} \langle \dot{\sigma}_0 \rangle = & \Gamma_R \sin^2 \left(\frac{\theta}{2} \right) \frac{1}{2} (\langle I_2 \rangle + \langle \sigma_z \rangle) \\ & + \Gamma_R \cos^2 \left(\frac{\theta}{2} \right) \frac{1}{2} (\langle I_2 \rangle - \langle \sigma_z \rangle) - \Gamma_L \langle \sigma_0 \rangle, \end{aligned} \quad (\text{B5})$$

$$1 = \langle I_2 \rangle + \langle \sigma_0 \rangle, \quad (\text{B6})$$

where σ_i are the usual Pauli spin matrices in the eigenbasis $\{|e\rangle, |g\rangle\}$, $I_2 \equiv |e\rangle\langle e| + |g\rangle\langle g|$, and $\sigma_0 \equiv |0\rangle\langle 0|$. We are only interested in the steady-state solutions.

Integrating out σ_0 we obtain the following equations in the steady state:

$$0 = -i\Delta_{cl}\langle a \rangle_{ss} - \frac{\kappa}{2}\langle a \rangle_{ss} - ig\langle \sigma_- \rangle_{ss}, \quad (\text{B7})$$

$$0 = -(i\Delta_{ql} + \gamma_d)\langle \sigma_- \rangle_{ss} + ig\langle a\sigma_z \rangle_{ss}, \quad (\text{B8})$$

$$\begin{aligned} 0 = & -\gamma_r(\langle I_2 \rangle_{ss} + \langle \sigma_z \rangle_{ss}) + \gamma_p(\langle I_2 \rangle_{ss} \\ & - \langle \sigma_z \rangle_{ss}) + 2ig(\langle a^\dagger \sigma_- \rangle_{ss} - \langle a \sigma_+ \rangle_{ss}), \end{aligned} \quad (\text{B9})$$

$$\langle I_2 \rangle_{ss} = \frac{1 + \langle \sigma_z \rangle_{ss} \cos \theta \Gamma_R / (2\Gamma_L)}{1 + \Gamma_R / (2\Gamma_L)}, \quad (\text{B10})$$

where $\gamma_r \equiv \gamma_\downarrow + \Gamma_R \sin^4(\theta/2)$, $\gamma_p \equiv \gamma_\uparrow + \Gamma_R \cos^4(\theta/2)$, and $\gamma_d \equiv 2\gamma_\phi + (\gamma_\uparrow + \gamma_\downarrow + \Gamma_R)/2$.

Introducing the last equation into the third equation we obtain

$$\begin{aligned} \langle \dot{\sigma}_z \rangle_{ss} = 0 = & \gamma(W_p - \langle \sigma_z \rangle_{ss}) \\ & + 2ig(\langle a^\dagger \sigma_- \rangle_{ss} - \langle a \sigma_+ \rangle_{ss}), \end{aligned} \quad (\text{B11})$$

$$\gamma \equiv \gamma_p + \gamma_r - (\gamma_p - \gamma_r) \frac{\cos \theta \Gamma_R / (2\Gamma_L)}{1 + \Gamma_R / (2\Gamma_L)}, \quad (\text{B12})$$

$$W_p \equiv \frac{1}{\gamma} \frac{\gamma_p - \gamma_r}{1 + \Gamma_R / (2\Gamma_L)}. \quad (\text{B13})$$

So far, no approximations have been made.

We need to solve the following equations, known as the Maxwell-Bloch equations:

$$0 = -i\Delta_{cl}\langle a \rangle_{ss} - \frac{\kappa}{2}\langle a \rangle_{ss} - ig\langle \sigma_- \rangle_{ss}, \quad (\text{B14})$$

$$0 = -(i\Delta_{ql} + \gamma_d)\langle \sigma_- \rangle_{ss} + ig\langle a\sigma_z \rangle_{ss}, \quad (\text{B15})$$

$$0 = \gamma(W_p - \langle \sigma_z \rangle_{ss}) + 2ig(\langle a^\dagger \sigma_- \rangle_{ss} - \langle a \sigma_+ \rangle_{ss}). \quad (\text{B16})$$

Although the above equations do not contain the third level, the trace of the density matrix is not 1. This information is contained in the value of I_2 which we also eliminated in the above. Therefore, the above equations describe the equations

of motion of a two-level system, and can be obtained from the following two-level master equation:

$$\begin{aligned} \dot{\rho} = & -i[H, \rho] + \Gamma_\downarrow \mathcal{D}(\sigma_-) + \Gamma_\uparrow \mathcal{D}(\sigma_+) + \Gamma_\phi \mathcal{D}(\sigma_z) + \kappa \mathcal{D}(a), \\ H = & \frac{\Omega}{2} \sigma_z + g(\sigma_+ a + \sigma_- a^\dagger) + \omega_c a^\dagger a \\ & + i\sqrt{\frac{\kappa}{2}} E \cos(\omega_d t)(a^\dagger - a), \end{aligned} \quad (\text{B17})$$

with effective pumping, relaxation, and pure dephasing rates given by

$$\Gamma_\uparrow = \gamma(1 + W_p)/2, \quad (\text{B18})$$

$$\Gamma_\downarrow = \gamma(1 - W_p)/2, \quad (\text{B19})$$

$$\Gamma_\phi = (\gamma_d - \gamma)/2. \quad (\text{B20})$$

Notice that the two-level system reduction is exact when determining the steady-state expectation values of operators and correctly describes the time evolution of the photon field when $\kappa \ll \Gamma_{\downarrow, \uparrow, \phi}$, when the DQD will adiabatically follow the photon field. To determine values of experimentally relevant operators, we express the operators in terms of the three-level system operators, then use Eqs. (B10) and (B6) to express the operators in terms of two-level system operators obtained from solving (B17).

2. Semiclassical approximation

Notice that the system of equations (B14) is not closed. In fact we can never obtain a closed system of equations using this method. If we were to find the equations of motion for quantities like $\langle a\sigma_z \rangle$ we would obtain equations that depend on other new terms and so on. Since the master equation cannot be solved exactly there are two main approximations that are usually employed at this stage. First, one can do a semiquantum approximation which assumes $\langle \sigma_z a \rangle \approx \langle \sigma_z \rangle \langle a \rangle$. An even stronger approximation, which we will use, is the semiclassical approximation which assumes that the cavity field a resembles a classical field. This allows us to factorize operator expectation values such that $\langle aO \rangle \approx \langle a \rangle \langle O \rangle$ for any DQD operator O . In this approximation the equations of motion become

$$0 = -i\Delta_{cl}\langle a \rangle_{ss} - \frac{\kappa}{2}\langle a \rangle_{ss} - ig\langle \sigma_- \rangle_{ss}, \quad (\text{B21})$$

$$0 = -(i\Delta_{ql} + \gamma_d)\langle \sigma_- \rangle_{ss} + ig\langle a \rangle_{ss} \langle \sigma_z \rangle_{ss}, \quad (\text{B22})$$

$$0 = \gamma(W_p - \langle \sigma_z \rangle_{ss}) + 2ig(\langle a^\dagger \rangle_{ss} \langle \sigma_- \rangle_{ss} - \langle a \rangle_{ss} \langle \sigma_+ \rangle_{ss}). \quad (\text{B23})$$

The last two equations imply

$$\langle \sigma_z \rangle_{ss} = \frac{W_p}{1 + G|\langle a \rangle_{ss}|^2}, \quad (\text{B24})$$

$$\langle \sigma_- \rangle_{ss} = ig|\langle a \rangle_{ss}| \frac{\langle \sigma_z \rangle_{ss}}{i\Delta_{ql} + \gamma_d}, \quad (\text{B25})$$

where $G = 4g^2\gamma_d/[\gamma(\gamma_d^2 + \Delta_{ql}^2)]$. Introducing this back into the EOM for the cavity field we obtain the following equation:

$$\langle a \rangle_{ss} \left(1 - \frac{1}{\frac{\kappa}{2} + i\Delta_{cl}} \frac{g^2}{i\Delta_{ql} + \gamma_d} \frac{W_p}{1 + G|\langle a \rangle_{ss}|^2} \right) = 0.$$

Equating the imaginary parts on both sides of the equation we obtain the so-called line pulling formula which determines the lasing frequency ω_l :

$$\omega_l = \omega_c \frac{\gamma_d}{\gamma_d + \kappa/2} + \Omega \frac{\kappa/2}{\kappa/2 + \gamma_d}. \quad (\text{B26})$$

The above equation implies that the lasing frequency will be “pulled” towards the frequency of the higher quality part of the system.

Analyzing Eq. (B26) we see that there is a critical W_p which yields qualitatively different solutions. When $W_p < 2\kappa/(G\gamma)$ only the trivial solution $|\langle a \rangle_{ss}|^2 = 0$ is allowed. However, when $W_p > 2\kappa/(G\gamma)$ another more interesting solution is allowed. This critical W_p is known as the laser threshold $W_{th} \equiv 2\kappa/(G\gamma)$.

Therefore, we express the cavity field intensity as

$$|\langle a \rangle_{ss}|^2 = \begin{cases} 0, & W_p < W_{th}, \\ \frac{1}{G} \left(\frac{W_p}{W_{th}} - 1 \right), & W_p > W_{th}. \end{cases} \quad (\text{B27})$$

The steady-state inversion is

$$\langle \sigma_z \rangle_{ss} = \begin{cases} W_p, & W_p < W_{th}, \\ W_{th}, & W_p > W_{th}. \end{cases} \quad (\text{B28})$$

We can also obtain the steady-state current:

$$\begin{aligned} I_{ss} &\equiv \Gamma_R \rho_{RR} = \frac{\Gamma_R}{2[1 + \Gamma_R/(2\Gamma_L)]} (1 - \langle \sigma_z \rangle_{ss} \cos \theta) \\ &\quad + \Gamma_R \sin \theta \operatorname{Re}(\langle \sigma_- \rangle_{ss}) \\ &= \frac{\Gamma_R}{2[1 + \Gamma_R/(2\Gamma_L)]} (1 - \langle \sigma_z \rangle_{ss} \cos \theta), \end{aligned} \quad (\text{B29})$$

where we used the fact that $\langle \sigma_- \rangle_{ss} = 0$ due to phase diffusion as we subsequently show in (B34).

3. Transmission

We have found the expressions for the cavity field intensity and inversion. At this point we reintroduce the classical perturbative probing field $E \cos(\omega_d t)$ into the initial equations and calculate the other system operators keeping in mind that inversion and field intensity are fixed due to nonperturbative processes. According to linear response theory the perturbation E will introduce an oscillatory perturbation rotating at

the driving frequency ω_d such that the steady-state cavity field is

$$\lim_{t \rightarrow \infty} a(t) = a_l e^{-i\omega_l t} + a_d e^{-i\omega_d t}, \quad (\text{B30})$$

where a_l is the field operator in the absence of the driving field E , while a_d is a small perturbation induced by the driving field. The DQD coherence σ_- will be similarly affected such that

$$\lim_{t \rightarrow \infty} \sigma_-(t) = \sigma_-^l e^{-i\omega_l t} + \sigma_-^d e^{-i\omega_d t}. \quad (\text{B31})$$

In this case the equations of motion for $\langle a_l \rangle$ are

$$\langle \dot{a}_l \rangle_{ss} = 0 = -i\Delta_{cl} \langle a_l \rangle_{ss} - \frac{\kappa}{2} \langle a_l \rangle_{ss} - ig \langle \sigma_-^l \rangle_{ss}, \quad (\text{B32})$$

$\langle \dot{\sigma}_-^l \rangle_{ss} = 0 = -(i\Delta_{ql} + \gamma_d) \langle \sigma_-^l \rangle_{ss} + ig \langle a_l \rangle_{ss} \langle \sigma_z \rangle_{ss}$, (B33) where the inversion $\langle \sigma_z \rangle_{ss}$ was calculated in (B28). It's easy to solve the above equations of motion and obtain

$$\langle a_l \rangle_{ss} = \langle \sigma_-^l \rangle_{ss} = 0. \quad (\text{B34})$$

This effect is known as phase diffusion [26]. The equations of motion for $\langle a_d \rangle$ are

$$\langle \dot{a}_d \rangle_{ss} = 0 = -i\Delta_{cd} \langle a_d \rangle_{ss} - \frac{\kappa}{2} \langle a_d \rangle_{ss} - ig \langle \sigma_-^d \rangle_{ss} + \sqrt{\kappa/8} E, \quad (\text{B35})$$

$$\langle \dot{\sigma}_-^d \rangle_{ss} = 0 = -(i\Delta_{qd} + \gamma_d) \langle \sigma_-^d \rangle_{ss} + ig \langle a_d \rangle_{ss} \langle \sigma_z \rangle_{ss}, \quad (\text{B36})$$

where, again, the inversion $\langle \sigma_z \rangle_{ss}$ was calculated in (B28) and is not influenced by the perturbation E . By defining the observable $A \equiv \sqrt{2\kappa} \langle a_d \rangle_{ss} / E$ we can solve the above to obtain

$$A = \frac{\kappa/2}{i\Delta_{cd} + \kappa/2 - g^2 \langle \sigma_z \rangle_{ss} / (i\Delta_{qd} + \gamma_d)}. \quad (\text{B37})$$

Notice that $|A|^2$ gives the transmission coefficient of a microwave of strength E through the transmission line resonator, while $\phi = \arctan[I(A)/\operatorname{Re}(A)]$ gives the phase shift of the microwave. Although the above formula looks simple it has a complicated dependence on detuning since g , Δ_{qd} , Δ_{cd} , ω_d , and σ_z are all functions of detuning ϵ .

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