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Quantum phase transition of light in the Rabi–Hubbard model

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Abstract

We discuss the physics of the Rabi–Hubbard model describing large arrays of coupled cavities interacting with two level atoms via a Rabi nonlinearity. We show that the inclusion of counter-rotating terms in the light–matter interaction, often neglected in theoretical descriptions based on Jaynes–Cumming models, is crucial to stabilize finite-density quantum phases of correlated photons with no need for an artificially engineered chemical potential. We show that the physical properties of these phases and the quantum phase transition occurring between them is remarkably different from those of interacting bosonic massive quantum particles. The competition between photon delocalization and Rabi nonlinearity drives the system across a novel Z_2 parity symmetry-breaking quantum phase transition between two gapped phases, a Rabi insulator and a delocalized super-radiant phase.

(Some figures may appear in colour only in the online journal)

1. Introduction

Recent years have seen an enormous increase in the level of control over light–matter interactions at the quantum level with both atomic and solid-state systems [1–3]. These major experimental achievements have brought forth a novel class of systems where light and matter play equally important roles in emergent quantum many body phenomena. The basic building block of such systems is the elementary cavity QED (CQED) system formed by a two-level system (TLS) interacting with a single mode of an electromagnetic resonator.

When CQED systems are coupled to form a lattice a plethora of novel phenomena are expected to emerge. The possibility of quantum phase transitions of light between Mott-like insulating and superfluid phases [4, 5], induced by the competition between photon blockade [6–8] and inter-cavity photon tunnelling has stimulated a great deal of discussion [9–13]. More recently proposals appeared to realize artificial gauge fields [14], photonic lattices with non-trivial band topology supporting chiral edge modes [15, 16] and quantum Hall fluids of light [17, 18, 19].

A major challenge on the way towards quantum simulation with photons arises due to the very nature by

which photons interact with their environment and with the matter field. In contrast to bosonic massive particles, photons can be annihilated by creating matter-like excitations or by exciting modes of their electromagnetic environment, hence their number is not conserved. To describe this situation for a photon gas one says that photons have zero chemical potential [20]. As a consequence, in order to engineer non-trivial quantum many body states other than the vacuum [21] one would require an external non-equilibrium drive to balance the unavoidable losses. While the intrinsic non-equilibrium nature of these systems [22] represent a major source of excitement that fits nicely with the recent interest in the physics of strongly interacting quantum systems in different non-equilibrium regimes, the possibility to have non-trivial equilibrium physics in a model of photons and atoms that could be potentially engineered in a lab represents an interesting perspective, worth to be pursued.

In this paper, by going back to the basic elementary CQED Hamiltonian, the Rabi model, we investigate the equilibrium phase diagram of a lattice of CQED systems. In particular we will show that the inclusion of counter-rotating terms in the light–matter interaction, often neglected in theoretical descriptions based on Jaynes–Cumming (JC)

models, is crucial to stabilize finite-density quantum phases of correlated photons out of the vacuum, with no need for an artificially engineered chemical potential, the role of which can be effectively played by the light–matter interaction strength. As a result of an interaction term that explicitly breaks the conservation of total number of excitations, the quantum phases and the phase transition occurring between them turn to be remarkably different from those of massive bosonic particles. The competition between photon delocalization and Rabi nonlinearity drives the system across a novel Z_2 parity symmetry-breaking quantum phase transition between two gapped phases, a Rabi insulator and a delocalized super-radiant phase where the TLSs polarize to generate a ferroelectrically ordered state and the photon coherence acquires a non-vanishing expectation value due to hopping. This novel quantum criticality shares some similarity with the Dicke phase transition of quantum optics, with the addition of non-trivial dynamical and spatial quantum fluctuations, and turns out to be in the universality class of the Ising model. We draw a complete picture of this quantum phase transition, that we have studied in a recent paper [23], including the calculation of the response to a weak local driving which allows to probe the photon spectral function. We note that a similar model with Rabi nonlinearity, although still in the presence of an artificial chemical potential, has been recently discussed in the literature [24]. The results of this study qualitatively agree with ours, even though their consequences on the universality class of the quantum phase transition have not been fully discussed.

The paper is organized as follows. In section 2 we introduce the Rabi model describing a single photonic mode coupled to a TLS. In section 3 we move to a tight-binding lattice model of coupled resonators with Rabi nonlinearity. We discuss the nature of the quantum phase transition in this model, present the Gutzwiller mean-field phase diagram and an effective low-energy description in terms of an Ising spin model in a transverse field that allows to access the spectrum of excitations. Section 4 is devoted to discussions and conclusions.

2. Single resonator limit: the Rabi model

We consider a single CQED unit with a TLS coupled to a radiation field. The basic Hamiltonian, introduced by Rabi [1, 25], reads

$$\mathcal{H}_R = \omega_0 a^\dagger a + \omega_q \sigma^+ \sigma^- + g(a^\dagger + a)(\sigma^+ + \sigma^-). \quad (1)$$

In the following we introduce the detuning δ between the resonator and the TLS

$$\delta = \omega_q - \omega_0$$

and mainly focus on the resonant case ($\delta = 0$) unless explicitly stated. The light–matter interaction contains both rotating and counter-rotating terms which do not conserve the total number of excitations, $\mathcal{N} = a^\dagger a + \sigma^+ \sigma^-$. These are often discarded, within the so called rotating-wave approximation, to obtain the celebrated JC Hamiltonian, that we write below for later convenience.

$$\mathcal{H}_{JC} = \omega_0 a^\dagger a + \omega_q \sigma^+ \sigma^- + g(a^\dagger \sigma^- + \sigma^+ a). \quad (2)$$

The presence of counter-rotating terms has some important consequences, that we are now going to discuss. The $U(1)$ symmetry associated with conservation of number of excitations breaks down to a discrete Z_2 symmetry generated by the parity operator $\mathcal{P}$ defined as

$$\mathcal{P} = e^{i\pi \mathcal{N}} \quad \mathcal{N} = a^\dagger a + \sigma^+ \sigma^- = a^\dagger a + \frac{1}{2}(\sigma_z + 1). \quad (3)$$

It is easy to see that the action of the parity on the photonic degrees of freedom is

$$\mathcal{P}^\dagger a \mathcal{P} = -a \quad \mathcal{P}^\dagger a^\dagger \mathcal{P} = -a^\dagger \quad (4)$$

while for the TLS we have

$$\mathcal{P}^\dagger \sigma_x \mathcal{P} = -\sigma_x \quad \mathcal{P}^\dagger \sigma_z \mathcal{P} = \sigma_z. \quad (5)$$

We therefore conclude that the action of the parity operator leaves the Rabi Hamiltonian invariant

$$\mathcal{P}^\dagger \mathcal{H}_R \mathcal{P} = \mathcal{H}_R \quad \longrightarrow \quad [\mathcal{P}, \mathcal{H}_R] = 0. \quad (6)$$

While the discrete Z_2 symmetry prevents a full closed-form solution, the Rabi model (1) has recently been shown to be nevertheless integrable [26] and its spectrum written in implicit form. In order to get more physical intuition into the nature of this solution, we proceed here by exact numerical diagonalization of the Rabi model by introducing a cut-off on the maximum number of photons $N_{\max}$ to truncate its Hilbert space.

Let us now come back to the role of the Z_2 symmetry. As the number of excitations is not conserved the exact eigenstates of the Rabi model will be labelled only by the parity quantum number (in addition to energy). This immediately implies, by symmetry, that the ground state of the single-site Rabi model will have, for any g/ω_0 ,

$$\langle a \rangle = 0 \quad \langle \sigma_x \rangle = 0 \quad (7)$$

since both operators are odd under Z_2 . In addition, as the ground state will be a linear combination of states with different number of excitations, the average number of photons in the Rabi ground state will be different from zero and, more specifically, a smoothly increasing function of g/ω_0 , even in absence of an external driving. This is indeed confirmed by the exact numerical solution of the model. In figure 1 we plot the average number of excitations $\langle \mathcal{N} \rangle$ and the average value of the photon operator $\langle a \rangle$ (the TLS $\langle \sigma_x \rangle$ behaves similarly) as a function of g/ω_0 . We notice the former increases smoothly with g/ω_0 . Similarly, the order parameters of the Z_2 symmetry are zero, as the ground state has always a well defined parity. This comes with no surprises, as the single-site Rabi model involves just a single atomic and a photonic mode that cannot result in symmetry breaking (the model is zero dimensional). It is worth in this respect to stress the difference with the more often discussed Dicke model [27], where a large number N of TLS are coupled to a single photonic mode and features in the thermodynamic limit $N \rightarrow \infty$ a genuine spontaneous symmetry breaking of the parity.

In addition to being populated by a finite number of excitations and symmetric, the ground state of the Rabi Hamiltonian is also squeezed, i.e. $\langle a^2 \rangle \neq 0$. In particular, due to the discrete nature of the parity symmetry, one can have for the Rabi ground state that $\langle (a^{2m}) \rangle \neq 0$ since by creating

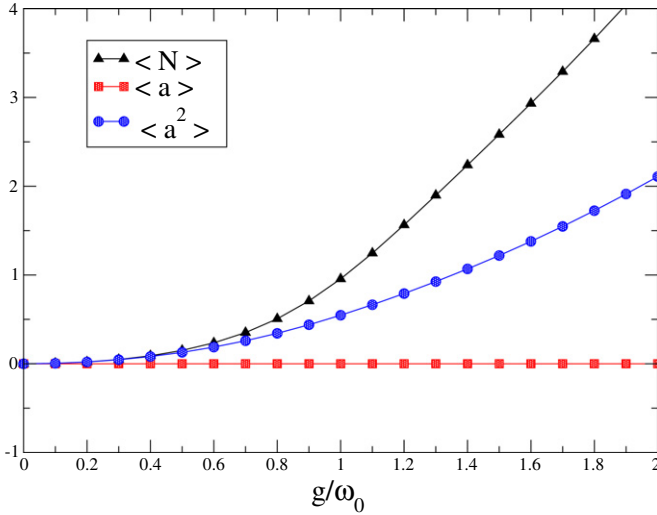


Figure 1. Ground state properties of the single-site Rabi model. We plot the average number of excitations in the ground state $\langle \mathcal{N} \rangle$, the average value of the photon field in the resonator $\langle a \rangle$ and the average squeezing $\langle a^2 \rangle$ as a function of g/ω_0 , at zero detuning $\delta = 0$.

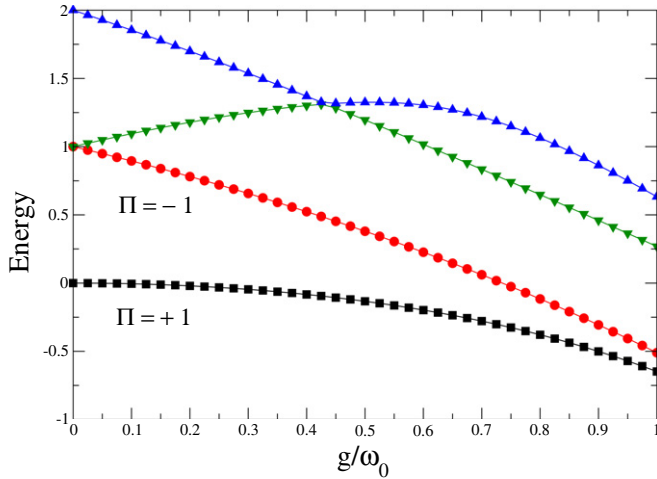


Figure 2. Low-lying spectrum of the single-site Rabi model. We plot as a function of the light-matter interaction g/ω_0 and at zero detuning $\delta = 0$ the ground state and the first few excited states, labelled by the parity $\Pi = \pm$. We notice that the ground state and the first excited state are almost degenerate for large g/ω_0 and that the former has always a fixed parity.

an even number of photons the overall parity is not affected. This property will play an important role when discussing the critical behaviour of the lattice Rabi model. We now discuss the structure of the low-lying excited states of the Rabi model. In figure 2, we plot the four lowest energy levels of the Rabi model at zero detuning as a function of g/ω_0 . We notice, quite interestingly, that (i) the ground state of the Rabi model remains an even parity state for any g/ω_0 , i.e. no level crossing between the ground state and the first excited state occurs as a function of g/ω_0 , and (ii) the ground state and the first excited state are quasi-degenerate in the ultra-strong coupling regime $g \gtrsim \omega_0$ with an exponentially small energy splitting Δ , while the gap to the next energy level stays of order one (see figure 3). Since this latter observation will play a crucial

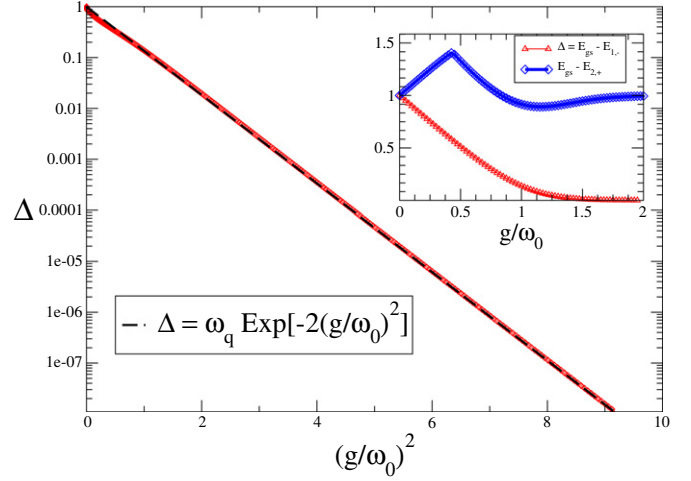


Figure 3. Splitting between the ground state and the first excited state in the single-site Rabi model. The red line is the exact numerical solution, the dashed black line is the asymptotic expression $\Delta = \omega_q e^{-2(g/\omega_0)^2}$. In the inset we plot the gap to the second excited state, which remains of order one even at large coupling $g \gg \omega_0$.

role later on in constructing an effective low-energy model for the Rabi lattice model, in the next subsection we give a simple analytical explanation for the exponential splitting that agrees well with the numerical result.

2.1. Splitting in the ultra-strong coupling regime

Here we want to discuss the single-site Rabi model in the limit $g/\omega_0 \gg 1$. From the exact solution we notice that the ground state and the first excited state are almost degenerate in this ultra-strong coupling regime (see figure 2) and with opposite parity. Here we want to get additional insight into this quasi-degeneracy and in particular to compute the splitting Δ as a function of g/ω_0 . Let us start from the Rabi Hamiltonian that we rewrite for simplicity here

$$\mathcal{H} = \omega_0 a^\dagger a + \frac{\omega_q}{2} \sigma_z + g(a^\dagger + a)\sigma_x. \quad (8)$$

For $\omega_q = 0$ the model can be trivially solved. Indeed in this case σ_x is conserved, namely we can define two Hamiltonians

$$\mathcal{H}_\pm = \omega_0 a^\dagger a \pm g(a^\dagger + a) \quad (9)$$

representing two displaced harmonic oscillators. Let us define as $\varphi_\pm(x)$ the ground states of these two Hamiltonians, corresponding respectively to $\pm g$. It is clear that the ground state of the original $\mathcal{H}$ for $\omega_q = 0$ is doubly degenerate as the two states

$$|\Psi_\pm\rangle = \varphi_\pm(x)|\sigma_x\rangle \quad |\sigma_x\rangle = |\rightarrow\rangle, |\leftarrow\rangle \quad (10)$$

are exactly degenerate. Indeed the ground state energy of the displaced harmonic oscillator reads

$$E_\pm = \frac{\omega_0}{2} - \frac{g^2}{\omega_0} \quad (11)$$

independent of the sign of g . Using the two above degenerate states we can construct two new states with well defined parity:

$$|\pm\rangle_0 = \frac{1}{\sqrt{2}}(\varphi_+(x)|\rightarrow\rangle \pm \varphi_-(x)|\leftarrow\rangle). \quad (12)$$

It is easy to see that under a parity transformation

$$\mathcal{P}|+\rangle_0 = |+\rangle_0 \quad \mathcal{P}|-\rangle_0 = -|-\rangle_0. \quad (13)$$

Any finite ω_q gives quantum mechanical fluctuations to σ_x and results into a lifting of this degeneracy. The perturbation

$$V_{\sigma_z} = \frac{\omega_q}{2} \sigma_z \quad (14)$$

results in a splitting

$$\begin{aligned} \Delta &= \frac{\omega_q}{2} [\langle +|\sigma_z|+\rangle_0 - \langle -|\sigma_z|-\rangle_0] \\ &= \omega_q \int dx \varphi_+(x) \varphi_-(x) = \omega_q e^{-2g^2/\omega_0^2}. \end{aligned} \quad (15)$$

The agreement between this result and the exact numerics in the strong coupling regime $g/\omega_0 \gg 1$ is remarkable as shown in figure 3.

3. The Rabi–Hubbard model

We now consider a model of localized photonic modes, $a_{\mathbf{R}}, a_{\mathbf{R}}^\dagger$, hopping at a rate J on a lattice with coordination z , and locally coupled by a Rabi interaction (1) to a set of a TLSs, described by Pauli operators $\sigma_{\mathbf{R}}^\pm$. The Hamiltonian for this interacting photonic lattice model that we refer to as the Rabi–Hubbard (RH) model reads

$$\mathcal{H} = - \sum_{(\mathbf{R}\mathbf{R}')} J_{\mathbf{R}\mathbf{R}'} x_{\mathbf{R}} x_{\mathbf{R}'} + \sum_{\mathbf{R}} H_{\mathbf{R}}[a_{\mathbf{R}}, a_{\mathbf{R}}^\dagger, \sigma_{\mathbf{R}}^x] \quad (16)$$

where $x_{\mathbf{R}} = a_{\mathbf{R}} + a_{\mathbf{R}}^\dagger$ while the single-site local Hamiltonian reads as

$$H_{\mathbf{R}}[a, a^\dagger, \sigma^\pm] = \omega_0 a^\dagger a + \omega_q \sigma^+ \sigma^- + g(a + a^\dagger) \sigma^x.$$

We note that in the above Hamiltonian we have included the counter-rotating terms in the hopping for the photon field at neighbouring sites.

Two trivial limits of the RH Hamiltonian (16) can be immediately discussed. In the limit of vanishing hopping, $J_{\mathbf{R}\mathbf{R}'} = 0$ all lattice sites decouple and the Hamiltonian reduces to a collection of single resonators with Rabi nonlinearity, whose physics we have discussed in the previous sections. In the opposite limit of vanishing nonlinearity $g = 0$ the photonic modes delocalize throughout the lattice and form a band. In the generic case $J \neq 0, g \neq 0$ the model cannot be solved exactly and a non-trivial behaviour is expected to emerge out of the competition between photon delocalization and Rabi nonlinearity [23].

The roots of this can be traced back to the single resonator limit. As discussed above, the counter-rotating terms leave the system with a discrete Z_2 symmetry associated to parity. Photon hopping in (16) can trigger a spontaneous breaking of this parity symmetry above some critical coupling $J_c(g)$, towards a phase where both $\langle a_{\mathbf{R}} \rangle \neq 0$ and $\langle \sigma_{\mathbf{R}}^x \rangle \neq 0$. As we are now going to show the RH model belongs to the Ising universality class and thus is fundamentally different from the JC one (see [12] and references therein). Instead, it can be seen as a delocalized super-radiant critical point reminiscent of the Dicke transition of quantum optics, an open version of which was recently realized, effectively, by coupling motional

degrees of freedom of an atomic BEC to a single mode of an optical cavity [27]. In contrast to the single-mode Dicke model however, which is an exact mean-field phase transition, the RH model displays non-trivial spatial and dynamical fluctuations. In the following we will discuss the qualitative physics of the RH model and its phase diagram using different analytical approaches.

3.1. Mean-field theory

Let's start deriving a mean-field theory for the RH model. Following the standard approach for bosons [28] we decouple the hopping term in (16) on a bond $\langle \mathbf{R}\mathbf{R}' \rangle$ as

$$H_{\text{hop}} = x_{\mathbf{R}} x_{\mathbf{R}'} \simeq x_{\mathbf{R}} \langle x_{\mathbf{R}'} \rangle + \langle x_{\mathbf{R}} \rangle x_{\mathbf{R}'} - \langle x_{\mathbf{R}'} \rangle \langle x_{\mathbf{R}} \rangle \quad (17)$$

so to reduce the original lattice Hamiltonian (16) to an effective single-site problem

$$\mathcal{H} \simeq \sum_{\mathbf{R}} H_{\mathbf{R}}^{\text{eff}}[a_{\mathbf{R}}, a_{\mathbf{R}}^\dagger, \sigma_{\mathbf{R}}^x] \quad (18)$$

with an effective Hamiltonian

$$H_{\mathbf{R}}^{\text{eff}} = \omega_r a_{\mathbf{R}}^\dagger a_{\mathbf{R}} + \omega_q \sigma_{\mathbf{R}}^+ \sigma_{\mathbf{R}}^- + g(a_{\mathbf{R}} + a_{\mathbf{R}}^\dagger) \sigma_{\mathbf{R}}^x + \psi_{\mathbf{R}}(a_{\mathbf{R}} + a_{\mathbf{R}}^\dagger) \quad (19)$$

describing a single-site Rabi model in a coherent driving field $\psi_{\mathbf{R}}$ that has to be self-consistently determined according to

$$\psi_{\mathbf{R}} = - \sum_{\mathbf{R}' \in v(\mathbf{R})} J_{\mathbf{R}\mathbf{R}'} \langle x_{\mathbf{R}'} \rangle_{\psi_{\mathbf{R}'}}. \quad (20)$$

Alternatively the effective field $\psi_{\mathbf{R}}$ can be determined by minimizing the total energy (at zero temperature)

$$E[\psi_{\mathbf{R}}] = \sum_{\mathbf{R}} \langle H_{\mathbf{R}}^{\text{eff}} \rangle_{\psi_{\mathbf{R}}} + \sum_{\mathbf{R}\mathbf{R}'} J_{\mathbf{R}\mathbf{R}'} \psi_{\mathbf{R}} \psi_{\mathbf{R}'}. \quad (21)$$

We notice the above mean-field approach is *exact* for infinite coordination lattices, i.e. in the limit $z \rightarrow \infty$, and can be seen as the first step of a systematic expansion in $1/z$. Quantum fluctuations beyond this Gutzwiller solution can be accounted for through the bosonic dynamical mean-field theory [29] or the quantum cavity method [30].

The mean-field phase boundary can be obtained, in the spirit of a Landau theory, by expanding the ground state energy (21) around the symmetric solution $\psi_{\mathbf{R}} = 0$. The phase transition towards a broken symmetry phase arises when the coefficient of the second order term becomes zero. This gives us the condition for the critical hopping J_c

$$\frac{1}{zJ_c} = \frac{1}{2} \int_0^\beta d\tau \langle T_\tau x(\tau) x(0) \rangle_{\text{loc}} = \frac{1}{2} C_{xx}(i\Omega = 0) \quad (22)$$

where we have defined the local bosonic correlation function in Matsubara frequencies as

$$C_{xx}(i\Omega) = \int_0^\beta d\tau e^{i\Omega\tau} \langle T_\tau x(\tau) x(0) \rangle_{\text{loc}}. \quad (23)$$

Using a representation in terms of exact eigenstates of the single-site Rabi Hamiltonian

$$\mathcal{H}_{\mathbf{R}}|n\rangle = E_n|n\rangle \quad (24)$$

we can write the expression for the mean-field phase boundary at zero temperature as

$$\frac{1}{zJ_c} = \sum_n |\langle n|x|\text{GS} \rangle|^2 \frac{1}{E_n - E_{\text{gs}}} \quad (25)$$

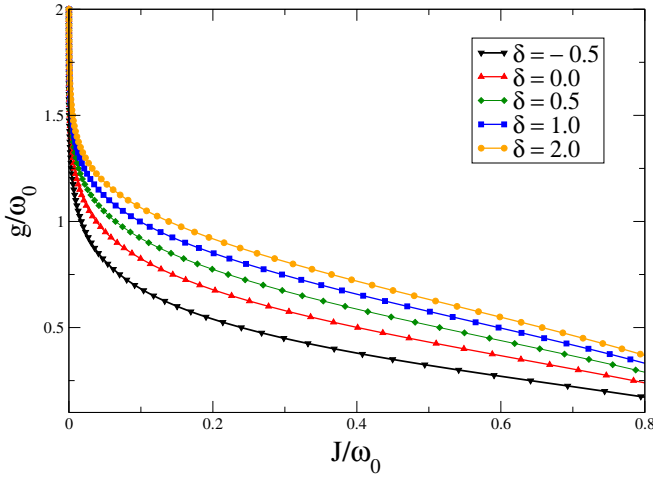


Figure 4. Mean-field phase diagram for the Rabi-Hubbard lattice model. We plot the critical value of the light-matter interaction g as a function of the hopping strength J , at different values of detuning $\delta = (\omega_q - \omega_0)/\omega_0$. Above a certain critical hopping $J_c(g, \delta)$ the Z_2 parity symmetry is spontaneously broken.

where $|\text{GS}\rangle$ is the ground state of the single-site Rabi model with energy E_{gs} . In figure 4 we plot the mean-field phase boundary in the J, g plane for different values of the detuning δ . We see that quite generically a disordered phase is stable until a critical hopping strength $J_c(g, \delta)$ is reached above which a broken symmetry phase emerges. This is characterized by the photon field becoming locally coherent, $\langle a_{\mathbf{R}} \rangle \neq 0$, and the TLS ordering $\langle \sigma_{\mathbf{R}}^x \rangle \neq 0$. Upon increasing the relative detuning between the TLS and the photon, $\delta = (\omega_q - \omega_0)/\omega_0$, from negative towards positive values the former will tend to fluctuate more strongly, as a consequence of a larger local transverse field $\omega_q \sigma^z$. This will favour local disordering, hence one may expect the size of the broken symmetry ordered phase to shrink, as indeed confirmed in figure 4.

Due to the discrete nature of the symmetry involved, a Z_2 associated to the parity, the qualitative features of the two phases and the transition between them are expected to be different, in terms of universal behaviour and physical properties, from the more often discussed JC lattice model [31] with its superfluid to Mott transition of polaritons. In particular, the Z_2 symmetry of the order parameter points towards an Ising universality class of the quantum phase transition separating two gapped phases, with a gap vanishing as a power-law at the critical point. In the next section we will confirm this scenario by introducing an effective low-energy theory for the RH model. This will allow us to compute the spectrum of low-lying excitations across the phase diagram and to evaluate the photon Green's function by including Gaussian fluctuations around the mean-field theory.

3.2. Effective low-energy description: Ising model in a transverse field

Here we discuss an effective low-energy description for the lattice Rabi model, which is valid asymptotically in the regime $g \gg \omega_0$. In this case we know from the exact solution of the single-site problem (see section 2) that the ground state and

the first excited state are almost degenerate and with opposite parity. This suggests to represent this low-energy doublet $|\pm\rangle$ as eigenstates of an effective local degree of freedom, a pseudo-spin 1/2 living on each site $\mathbf{R}$ of the lattice

$$\Sigma_{\mathbf{R}}^z |\pm\rangle = \pm |\pm\rangle. \quad (26)$$

Since the gap to the next energy level in the spectrum of the single-site Rabi model stays of order one at large g/ω_0 , this suggests building an effective Hamiltonian by projecting the local Hilbert space onto this low-energy doublet $|\pm\rangle$. The local Hamiltonian in this truncated basis reads, by definition

$$\mathcal{H}_R \simeq E_+ |+\rangle\langle +| + E_- |-\rangle\langle -| = \frac{\bar{E}}{2} \mathbb{I} + \frac{\Delta}{2} \Sigma^z \quad (27)$$

where $E_{\pm}$ are the two lowest eigenvalues of the Rabi spectrum, $\Delta = E_+ - E_- > 0$ the level splitting and $\bar{E}$ the average energy between the two levels. The latter contributes an irrelevant identity matrix that will be dropped in the following. Written in the pseudo-spin basis $|\pm\rangle$, the photon creation/annihilation $a_{\mathbf{R}}, a_{\mathbf{R}}^\dagger$ take a purely off-diagonal form because they do not couple states with the same parity.

$$\langle n | a | m \rangle = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} = \beta \Sigma^+ + \gamma \Sigma^-.$$

Similarly we get for the Hermitian conjugate $a^\dagger$

$$\langle n | a^\dagger | m \rangle = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix} = \gamma \Sigma^+ + \beta \Sigma^-.$$

Using these results we can now write the photon hopping as an effective spin-spin interaction between neighbouring sites. A straightforward calculation gives the following effective spin model Hamiltonian

$$\mathcal{H}_{\text{eff}} = - \sum_{\langle \mathbf{R}\mathbf{R}' \rangle} J_x \Sigma_{\mathbf{R}}^x \Sigma_{\mathbf{R}'}^x + \frac{\Delta}{2} \sum_{\mathbf{R}} \Sigma_{\mathbf{R}}^z \quad (28)$$

with $J_x = \frac{J}{2}(\gamma + \beta)^2$, where J is the hopping between the photonic resonators in the original RH model (16), that we take here between neighbouring sites only. The resulting Hamiltonian describes an Ising spin model in a transverse magnetic field which is known to display a quantum phase transition towards a Z_2 broken symmetry phase which is in the Ising universality class [32]. We note that the presence of a finite asymmetry between the rotating and counter-rotating terms in the photon hopping term would lead to an anisotropic spin-exchange $J_x \neq J_y$ which remains within the same Ising universality class. The parameters entering the effective model, J_x, Δ , depend strongly on the single-site physics and can be obtained numerically from the exact diagonalization of the Rabi Hamiltonian H_R . However their behaviour at strong light-matter coupling, $g \gg \omega_0$, can be obtained analytically. Indeed from the analysis of section 2 the level splitting was found to be exponentially suppressed at strong coupling, $\Delta = \omega_0 \exp[-2(g/\omega_0)^2]$. Similarly, one can write the photon field operator a in the basis of the low-energy doublet $|\pm\rangle$. To leading order we get for the off-diagonal terms $\langle - | a | + \rangle = \langle + | a | - \rangle \simeq g/\omega_0$ from which we conclude that for $g \gg \omega_0$ we have $J_x \simeq 2J(g/\omega_0)^2$ which we find to nicely match the numerical results obtained by numerical diagonalization [23].

3.2.1. Low-energy spectrum: mean field and fluctuations. In this section we analyse the low-energy excitations of the effective pseudo-spin-model equation (28) through the spin-wave theory. The classical ground state can be obtained by rotating each spin locally around Σ^y with a unitary operator

$$\Omega = \exp(i\theta \Sigma^y/2). \quad (29)$$

The angle θ is then fixed by minimizing the energy of a classical ferromagnetic state along x in the rotated basis, which gives the condition

$$\frac{\partial E_{\text{cl}}}{\partial \theta} = \cos \theta \left(zJ_x \sin \theta + \frac{\Delta}{2} \right) = 0. \quad (30)$$

This identifies two regimes, namely for $2zJ_x > \Delta$ we have

$$\sin \theta = -\frac{\Delta}{2zJ_x} \quad (31)$$

while for $2zJ_x < \Delta$ we have $\sin \theta = -1$, $\cos \theta = 0$. When expressed in terms of the hopping J between photons this identifies the mean-field phase boundary

$$zJ_c = \frac{\Delta}{(\beta + \gamma)^2}. \quad (32)$$

If we compute the average magnetization along x we find for $J > J_c$

$$\langle \Sigma_{\mathbf{R}}^x \rangle = \cos \theta = \sqrt{1 - (J_c/J)^2} \quad (33)$$

while $\langle \Sigma_{\mathbf{R}}^x \rangle = 0$ for $J < J_c$ from which we can conclude that at the classical level there is an Ising like phase transition at J_c between a quantum disordered paramagnet for $J < J_c$ and a ferromagnetically ordered state for $J > J_c$.

Once we have found the classically ordered ground state playing the role of the vacuum $|0\rangle$ we can now include the quantum fluctuations around this state at the leading (harmonic) order within a spin-wave approach. To do that we introduce the following representation of $\Sigma_{\mathbf{R}}^\alpha$ operators in terms of a set of harmonic oscillators

$$\Sigma_{\mathbf{R}}^x = 1 - \Pi_{\mathbf{R}} \quad (34)$$

$$\Sigma_{\mathbf{R}}^y = -\sqrt{2}p_{\mathbf{R}} \quad (35)$$

$$\Sigma_{\mathbf{R}}^z = \sqrt{2}x_{\mathbf{R}} \quad (36)$$

where $\Pi_{\mathbf{R}} = x_{\mathbf{R}}^2 + p_{\mathbf{R}}^2 - 1$ measure the strength of the quantum fluctuations, namely

$$\langle \Pi_{\mathbf{R}} \rangle = 0 \quad (37)$$

when evaluated on the vacuum (i.e. the classical ground state). We now substitute the expressions (34)–(36) in the effective Hamiltonian $\tilde{\mathcal{H}}_{\text{eff}}$ and keep only terms up to linear order in Π . Then, by using the condition (30) to cancel terms which are linear in the oscillators, we end up with a collection of harmonic oscillators describing the low-lying fluctuating modes around the classical ground state. The resulting Hamiltonian is then readily diagonalized in terms of bosonic modes $b_{\mathbf{q}}, b_{\mathbf{q}}^\dagger$

$$\tilde{H}_{\text{eff}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}} \quad (38)$$

where $\omega_{\mathbf{q}}$ describes the spectrum of low-lying excitations whose structure we now discuss as the hopping J is tuned

across the critical point. In the disordered phase $J < J_c$ we find

$$\omega_{\mathbf{q}} = \Delta \sqrt{(1 - \mathcal{J} \gamma_{\mathbf{q}})} \quad (39)$$

where we have introduced the reduced hopping $\mathcal{J} = J/J_c$, while for $J > J_c$ we get

$$\omega_{\mathbf{q}} = zJ(\beta + \gamma)^2 \sqrt{(1 - \gamma_{\mathbf{q}}/\mathcal{J})} \quad (40)$$

where we have assumed for concreteness a hypercubic lattice in $d = z/2$ dimensions,

$$\gamma_{\mathbf{q}} = \frac{1}{d} \sum_{a=1}^d \cos q_a \in [-1, 1], \quad (41)$$

with q_a the components of the wavevector $\mathbf{q}$.

The spectrum as expected is gapped on both sides of the transition

$$\omega_{\mathbf{q}} \simeq \omega_{\text{gap}}^\pm + \alpha_\pm |\mathbf{q}|^2 \quad (42)$$

with a gap vanishing in a power-law fashion at the transition,

$$\omega_{\text{gap}}^\pm = c_\pm |J - J_c|^{1/2}. \quad (43)$$

Right at the phase boundary $J = J_c$ the spectrum becomes gapless with a linear dispersion $\omega_{\mathbf{q}} = c |\mathbf{q}|$ as expected for an Ising quantum critical point, whose dynamical critical exponent is $z_{\text{dyn}} = 1$.

3.2.2. Response to a weak driving and photon spectral function. Here we want to use the effective model to discuss the response of the RH model to a weak local drive and to compute the associated response function, i.e. the photon spectral function. Let us imagine driving a single resonator at position $\mathbf{R}$ with a weak tone $\eta(t)$, i.e. to couple the system to the perturbation $V_{\text{drive}}(t) = \eta(t) x_{\mathbf{R}}$ and measure the field at the same port $x_{\mathbf{R}}(t)$. Using linear response theory we obtain

$$\langle x_{\mathbf{R}}(t) \rangle = \int dt' \chi_{\text{loc}}(t - t') \eta(t') \quad (44)$$

where the response function $\chi_{\text{loc}}(t)$ is the local retarded photon Green's function

$$\chi_{\text{loc}}(t > 0) = -i \sum_{\mathbf{q}} \langle [x_{\mathbf{q}}(t), x_{-\mathbf{q}}(0)] \rangle. \quad (45)$$

It is therefore interesting to compute this response function in the frequency domain and discuss the signatures of the Rabi quantum phase transition. This can be done using the effective spin model we have previously discussed. We start considering the imaginary time correlator

$$\chi(\mathbf{q}, \tau) = \langle T_\tau x_{\mathbf{q}}(\tau) x_{-\mathbf{q}}(0) \rangle \quad (46)$$

and then perform an analytic continuation.

It is easy to see that within our effective model we have $\chi(\mathbf{q}, \tau) \sim \langle T_\tau \Sigma_{\mathbf{q}}^x(\tau) \Sigma_{-\mathbf{q}}^x(0) \rangle$ where the average and the imaginary time evolution is done with respect to the spin Hamiltonian (28). The above correlation can be evaluated using our harmonic theory of fluctuations. In particular, after performing the rotation (29) and bosonizing the spin operator as in (34)–(36) we reduce the calculation to the evaluation of single particle correlators of harmonic oscillators. The result of this lengthy but straightforward calculation reads, in Matsubara frequencies $i\Omega_n = 2\pi i n T$

$$\chi(\mathbf{q}, i\Omega) \sim \cos^2 \theta A(\mathbf{q}, i\Omega) + \sin^2 \theta B(\mathbf{q}, i\Omega) \quad (47)$$

Here we have introduced the correlators

$$B(\mathbf{q}, i\Omega) = K_{\mathbf{q}} \left(\frac{1}{i\Omega + \omega_{\mathbf{q}}} - \frac{1}{i\Omega - \omega_{\mathbf{q}}} \right) \quad (48)$$

and

$$A(\mathbf{q}, i\Omega) = \langle x \rangle_0^2 \delta(i\Omega) + \left[\frac{K_{\mathbf{q}}^2}{4} + \frac{1}{4K_{\mathbf{q}}^2} - \frac{1}{2} \right] \times (n_B(\omega_{\mathbf{q}}) - n_B(-\omega_{\mathbf{q}})) \left(\frac{1}{i\Omega + 2\omega_{\mathbf{q}}} - \frac{1}{i\Omega - 2\omega_{\mathbf{q}}} \right) \quad (49)$$

with the bosonic occupation factor $n_B(\omega) = (e^{\beta\omega} - 1)^{-1}$, while $\langle x \rangle_0$ is the average Z_2 order parameter (including classical and quantum fluctuation contributions)

$$\langle x \rangle_0 = 2 \cos \theta \sum_{\mathbf{q}} \left[1 - \frac{1}{2} \left(K_{\mathbf{q}} + \frac{1}{K_{\mathbf{q}}} \right) \right]. \quad (50)$$

In addition we have introduced the kernel $K_{\mathbf{q}}$ which reads for $J < J_c$

$$K_{\mathbf{q}} = \frac{1}{\sqrt{1 - \mathcal{J}\gamma_{\mathbf{q}}}} \quad (51)$$

while in the ordered phase $J > J_c$

$$K_{\mathbf{q}} = \frac{1}{\sqrt{1 - \gamma_{\mathbf{q}}/\mathcal{J}}}. \quad (52)$$

With the above results at hand we now perform an analytic continuation onto the real axis $i\Omega \rightarrow \Omega + i\eta$ and sum over momenta to obtain the local retarded photon Green's function, whose imaginary part defines our spectral function

$$\text{Im } \chi_{\text{loc}}(\Omega) = \sum_{\mathbf{q}} \text{Im } \chi(\mathbf{q}, i\Omega \rightarrow \Omega + i\eta). \quad (53)$$

Let us discuss now these results at zero temperature $T = 0$ across the RH phase diagram. In the disordered phase, $J < J_c$, we have $\cos \theta = 0$ and the photon spectrum contains two quasi-particle peaks at $\Omega = \pm\omega_{\mathbf{q}}$, see equation (48). These give rise in the local spectrum to two broad high-energy features at frequencies $\Omega \in [\omega_-, \omega_+]$ and $\Omega \in [-\omega_+, -\omega_-]$ with edges $\omega_{\mp} = \Delta \sqrt{1 \mp \mathcal{J}}$. Those two RH bands are separated by a gap $\omega_{\text{gap}} = 2\omega_-$. For $J < J_c$ we obtain

$$\text{Im } \chi_{\text{loc}}(\Omega) = \theta(\Omega^2 - \omega_-^2) \theta(\omega_+^2 - \Omega^2) \frac{\Delta}{|\Omega|} \rho_0 \left(\frac{\Delta^2 - \Omega^2}{\Delta^2 - \omega_-^2} \right) \quad (54)$$

where $\rho_0(\omega)$ is the tight-binding density of state of the underlying photonic lattice. The top panel of figure 5 shows the photon spectral function in the Rabi insulating disordered phase. Right at the quantum critical point, $J = J_c$, the gap vanishes and the spectrum of excitations is gapless, $\omega_{\mathbf{q}} = c|\mathbf{q}|$. The photon spectral function can be obtained in closed form at this point and reads

$$\text{Im } \chi_{\text{loc}}(\Omega) = \theta(\omega_+^2 - \Omega^2) \frac{\Delta}{|\Omega|} \rho_0 \left(\frac{\Delta^2 - \Omega^2}{\Delta^2} \right). \quad (55)$$

Finally as the phase boundary is crossed and one enters the ordered phase at $J > J_c$ a new feature in the spectrum appears, related to the onset of a broken symmetry. In particular a sharp coherent delta peak at $\Omega = 0$ appears in the middle of a gap (see equation (49)) whose strength Z is controlled by the order parameter. In addition other quasi-particle peaks appear

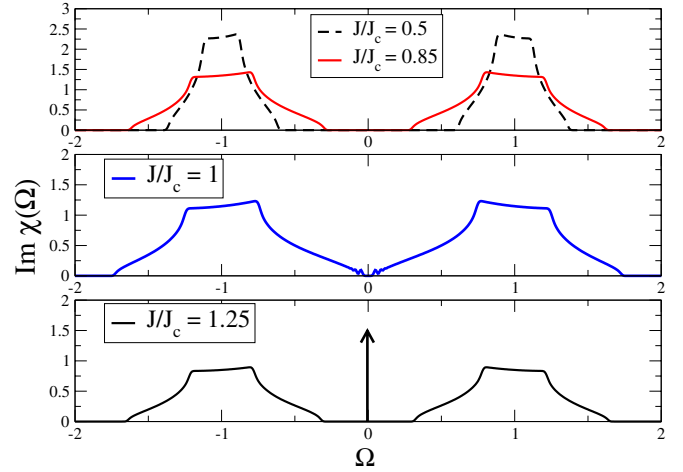


Figure 5. Imaginary part of the photon spectral function $\text{Im } \chi_{\text{loc}}(\Omega)$ for different values of the hopping strength across the critical point. In the insulating disordered phase $J < J_c$ the spectrum is gapped (top panel) and turns gapless at the quantum critical point $J = J_c$ with the spectrum vanishing as a power-law at low frequency. Finally in the ordered phase $J > J_c$ a gap opens and a sharp coherent peak emerges at $\Omega = 0$ signalling the breaking of the Z_2 symmetry.

in the spectral function at higher frequencies $\Omega = \pm 2\omega_{\mathbf{q}}$ which result into an additional contribution to the high-energy background of the local spectrum. As a result the ordered phase photon spectral function features two contributions, a coherent one at zero frequency and an incoherent term at higher frequencies

$$\text{Im } \chi_{\text{loc}}(\Omega) = Z\delta(\Omega) + \text{Im } \chi_{\text{loc}}^{\text{incoh}}(\Omega) \quad (56)$$

where the strength of the delta peak is given by

$$Z = [\mathcal{J}^2 - 1] \langle x \rangle_0^2.$$

The incoherent contribution is gapped and starts at the threshold $\omega_- \sim \sqrt{\mathcal{J} - 1}$. We plot the photon spectral function at the critical point as well as in the ordered phase in the lower panels of figure 5.

4. Discussions and conclusions

In this paper, we discuss the phase transition of light in a lattice of CQED systems. The basic building block of the lattice model we analyse here is the fundamental light-matter interaction Hamiltonian, the Rabi model. This model was previously discussed extensively by dropping the counter-rotating terms and introducing a chemical potential, the origin of which has been left unjustified—this is the Jaynes-Cummings-Hubbard model. We argue in the present manuscript that the neglected counterclockwise terms are relevant operators, changing completely the nature of the phase transition of these interacting light-matter systems. In particular, we show that the counterclockwise terms act as an effective chemical potential stabilizing, out of vacuum, finite-density quantum phases of correlated photons and material excitations.

This changes completely the nature of quantum criticality of lattice CQED systems from those based on the JC model.

While the latter is within the universality class of the Bose–Hubbard model, the RH model displays a Z_2 parity-breaking quantum phase transition, where the TLSs polarize to generate a ferroelectrically ordered state and the photon coherence acquires a non-vanishing expectation value due to hopping. This quantum phase transition shares some aspects of the Dicke super-radiant critical point, with the addition of non-trivial dynamical and quantum fluctuations that arise from the multi-mode nature of the photonic spectrum. From the point of view of its critical behaviour, the RH phase transition belongs to the universality class of the quantum Ising model as we have shown here by introducing a low-energy description, valid in the regime of large light–matter coupling, in terms of an emergent Z_2 low-energy doublet.

Interesting research directions that we leave for future work concern, for example, the role of dissipation and photon losses on the RH phase diagram, or the possible signatures of the underlying quantum criticality in terms of experimentally measurable quantities. In this respect, owing to the simple discrete nature of the Z_2 symmetry of the problem, a possible direction could involve the dynamical control of non-adiabatic crossing of the phase transition (such as the detuning, which acts as a transverse field for the TLS) and the observation of the resulting dynamics of defects formation, similar to what has been recently discussed in the context of trapped ion systems [33, 34]. Finally, a particularly appealing research direction is the designing of driven CQED architectures for realization of effective Hamiltonians with counter-rotating terms.

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References

- [1] Haroche S and Raimond J 2006 *Exploring the Quantum: Atoms, Cavities, and Photons* 1st edn (Oxford: Oxford University Press)
- [2] Yamamoto Y, Tassone F and Cao H 2000 *Semiconductor Cavity Quantum Electrodynamics* 1st edn (Berlin: Springer)
- [3] Schoelkopf R J and Girvin S M 2008 *Nature* **451** 664
- [4] Greentree A D, Tahan C, Cole J H and Hollenberg L C L 2006 *Nature Phys.* **2** 856
- [5] Hartmann M, Brandao F and Plenio M B 2006 *Nature Phys.* **2** 849
- [6] Tian L and Carmichael H J 1992 *Phys. Rev. A* **46** R6801
- [7] Imamoglu A, Schmidt H, Woods G and Deutsch M 1997 *Phys. Rev. Lett.* **79** 1467
- [8] Rebic S, Tan S M, Parkins A S and Walls D F 1999 *J. Opt. B: Quantum Semiclass. Opt.* **1** 490
- [9] Hartmann M, Brandao F and Plenio M 2008 *Laser Photon. Rev.* **2** 527
- [10] Tomadin A and Fazio R 2010 *J. Opt. Soc. Am. B* **27** A130
- [11] Houck A A, Türeci H E and Koch J 2012 *Nature Phys.* **8** 292
- [12] Schmidt S and Koch J 2013 *Ann. Phys., Berl.* **525** 395
- [13] Angelakis D G, Huo M, Kyoseva E and Kwek L C 2011 *Phys. Rev. Lett.* **106** 153601
- [14] Koch J, Houck A A, Hur K L and Girvin S M 2010 *Phys. Rev. A* **82** 043811
- [15] Hafezi M, Fan J, Migdall A and Taylor J 2013 arXiv:1302.2153
- [16] Petrescu A, Houck A A and Le Hur K 2012 *Phys. Rev. A* **86** 053804
- [17] Hafezi M, Lukin M D and Taylor J M 2013 *New J. Phys.* **15** 063001
- [18] Cho J, Angelakis D G and Bose S 2008 *Phys. Rev. Lett.* **101** 246809
- [19] Umucallar R O and Carusotto I 2011 *Phys. Rev. A* **84** 043804
- [20] Landau L D and Lifshitz E 1980 *Statistical Physics* 3rd edn, vol 5 (Washington, DC: Butterworth)
- [21] Klaers J, Schmitt J, Vewinger F and Weitz M 2010 *Nature* **468** 545
- [22] Houck A A, Türeci H E and Koch J 2012 *Nature Phys.* **8** 292–9
- [23] Schiró M, Bordyuh M, Öztóp B and Türeci H E 2012 *Phys. Rev. Lett.* **109** 053601
- [24] Zheng H and Takada Y 2011 *Phys. Rev. A* **84** 043819
- [25] Rabi I I 1937 *Phys. Rev.* **51** 652
- [26] Braak D 2011 *Phys. Rev. Lett.* **107** 100401
- [27] Baumann K, Guerlin C, Brennecke F and Esslinger T 2010 *Nature* **464** 1301
- [28] Fisher M P A, Weichman P B, Grinstein G and Fisher D S 1989 *Phys. Rev. B* **40** 546
- [29] Anders P, Gull E, Pollet L, Troyer M and Werner P 2010 *Phys. Rev. Lett.* **105** 096402
- [30] Krzakala F, Rosso A, Semerjian G and Zamponi F 2008 *Phys. Rev. B* **78** 134428
- [31] Hohenadler M, Aichhorn M, Schmidt S and Pollet L 2011 *Phys. Rev. A* **84** 041608
- [32] Sachdev S 2011 *Quantum Phase Transitions* 2nd edn (Cambridge: Cambridge University Press)
- [33] Pyka K, Keller J, Partner H L, Nigmatullin R, Burgermeister T, Meier D-M, Kuhlmann K, Retzker A, Plenio M B and Zurek W H 2012 arXiv:1211.7005
- [34] Ulm S, Roßnagel J, Jacob G, Degünther C, Dawkins S T, Poschinger U G, Nigmatullin R, Retzker A, Plenio M B and Schmidt-Kaler F 2013 arXiv:1302.5343