Negative electrohydrostatic pressure between superconducting bodies

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Despite being largely limited to bulk phenomena, well-known theoretical models of superconductivity like the Bardeen–Cooper–Schrieffer and Ginzburg–Landau theories have played a key role in the development of superconducting quantum devices. In this letter, we present a hydrodynamic non-relativistic scalar electrodynamic theory capable of describing systems comprising superconducting materials of arbitrary shape and apply it to predict the existence of a negative (attractive) pressure between planar superconducting bodies. For conventional superconductors with London penetration depth $\lambda_{\mathrm{L}} \approx 100$ nm, the pressure reaches tens of N/mm^2 at angstrom separations.

In conventional superconductors, steady-state bulk phenomena are accurately described by both the Bardeen-Cooper-Schrieffer (BCS) [1] and Ginzburg-Landau (GL) [2] theories. The former provides a microscopic origin for superconductivity via the phonon-mediated pairing of electrons into bosonic quasiparticles known as Cooper pairs, while the latter provides a phenomenological description of the resulting Bose-Einstein condensate [3] with a macroscopic order parameter representing its mean-field wave function. The two theories were shown to be equivalent near the superconducting critical temperature [4], and both reproduce the London theory [5]. Though the BCS theory is sufficiently general to predict time-dependent bulk phenomena, an effective macroscopic theory is desirable when such effects are triggered by electromagnetic sources in spatially inhomogeneous domains. To this end, generalized GL equations have been proposed to capture boundary and wave effects present in complex geometries [6], but a consensus has not been reached on their validity far below the critical temperature, a regime all too familiar to the burgeoning area of superconducting quantum devices [7].

In this letter, we present and explore predictions offered by a hydrodynamic representation of non-relativistic scalar electrodynamics applied to the superconducting order parameter at zero temperature. Few attempts have been made to solve this model's equations of motion (EOM) exactly [8], but simplified versions have been considered via relaxations of minimal coupling [9–12] and can be credited as the underpinning of Josephson phenomena and circuit quantum electrodynamics [13]. Such approximate descriptions of light-matter interactions have enabled coveted numerical analyses of superconducting circuits embedded in electromagnetic resonant structures [14, 15], but they rely on London-like boundary conditions between superconducting and non-superconducting domains that seem to harbor serious inconsistencies [16]. Our goal is not to provide a rigorous derivation of the theory (the literature contains some attempts [17, 18]), but rather to demonstrate that its unapproximated form circumvents spatial partitioning and implies a pressure between planar superconducting bodies that can be measured to determine its validity.

While our model shares similarities with the GL theory in that it describes the superconducting condensate with an order parameter, it differs in at least four important ways. First, in contrast to the diffusive time-dependent GL equations, our model entails wave-like dynamics implied by Schrödinger's equation. Second, we employ minimal coupling to all electromagnetic degrees of freedom, including the electric field via Gauss's law and Maxwell's correction to Ampere's law. Third, we incorporate arbitrary arrangements of both external drives and ionic backgrounds via normal (non-superconducting) source distributions. We take the latter to be static in nature, akin to the Jellium model of a metallic conductor [19], but generalizable to include dynamical fluctuations for effective descriptions of phononic excitations. Fourth, in considering regimes far below the critical temperature, we omit the self-interaction term that governs the GL phase transition. In our model, nonlinear phenomena arise instead from our more general treatment of light-matter interactions, and the Higgs mechanism that yields the condensate's equilibrium number density via spontaneous symmetry breaking of the $U(1)$ gauge group is replaced by requirement from the EOM that the bulk superconducting charge density cancels the prescribed ionic background.

Below, we present the Lagrangian and corresponding EOM at the heart of our model, along with an electrohydrodynamic representation of the Hamiltonian. Limiting our focus to electrostatic systems, we derive an electrohydrostatic condition arising from a self-consistent statement of Gauss's law and solve it numerically in the context of two planar superconducting bodies separated by vacuum. By considering variations in the system's electrohydrostatic energy with respect to the separation length, we find a negative (attractive) pressure between the two bodies that peaks at an emergent healing length. We conclude with a discussion of the length's significance.

Throughout the text, we employ the covariant formulation of electromagnetism with the Minkowski metric

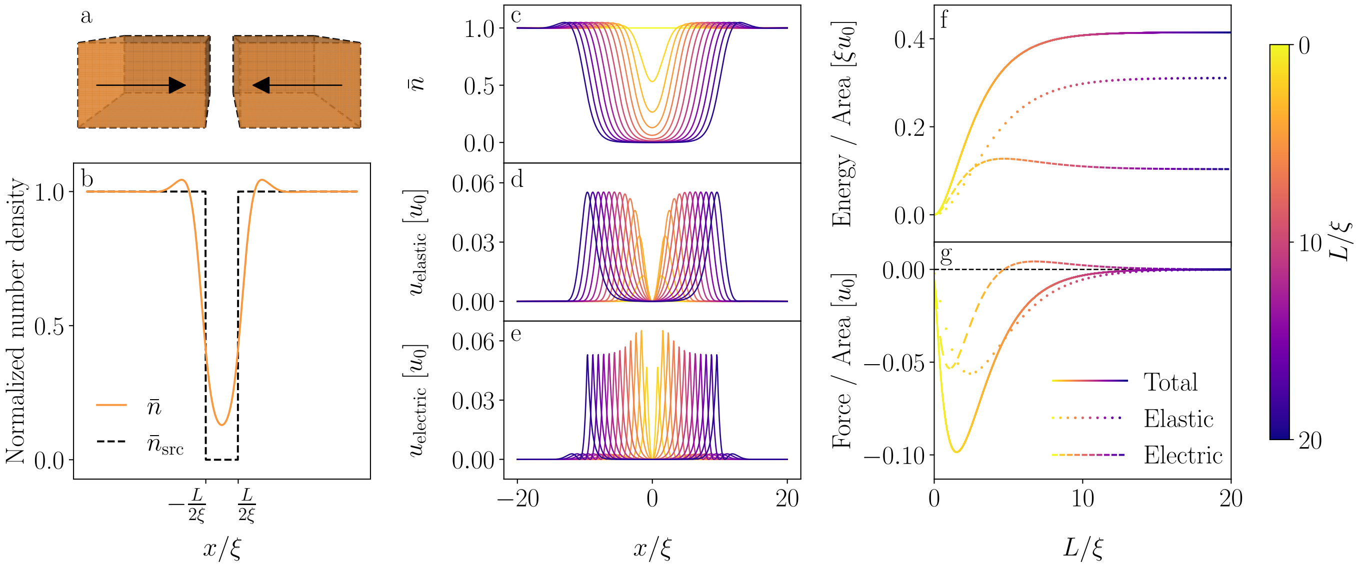

FIG. 1. The main result is summarized by a free body diagram (a) depicting the attractive force between two planar superconducting bodies. Calculation of this pressure begins with a numerical solution $\bar{n}$ to the electrohydrostatic condition sourced by two finitely separated ionic backgrounds $\bar{n}_{\mathrm{src}}$. An example solution with separation length $L = 4.8\xi$ is depicted in (b). We next calculate the electrostatic distribution of the normalized number density (c), elastic energy density (d), and electric energy density (e) for a range of separation lengths specified by the color bar. Energy densities are plotted in units of $u_0 = \hbar mc/\left(\mu_0 q^2 \lambda_{\mathrm{L}}^3\right)$ and are spatially integrated to find the energy per unit area (f) as a function of the separation length, whose negative derivative with respect to L yields the pressure (g). We note that the electric pressure changes sign at $L \approx 4.8\xi$.

$\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)^{\mu\nu}$, and we refer to the components of a four-vector as $X^\mu \equiv (X_0, \mathbf{X})^\mu$. Though the model describes non-relativistic charged superfluids, we find that a relativistic notation provides useful physical insight. We assume the effective Lagrangian governing the evolution of the order parameter $\psi \equiv \sqrt{n}e^{i\theta}$ and the electromagnetic four-potential A^μ is given by the non-relativistic theory of scalar electrodynamics under minimal light-matter coupling,

$$\mathcal{L} = \psi^* \left(i\hbar\frac{\partial}{\partial t} - qcA_0 - \frac{1}{2m}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\mathbf{A}\right)^2 \right)\psi - \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - A^\mu j_\mu, \tag{1}$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic tensor, j^ν is the four-current generated by normal charges, and q and m are the charge and mass of the superconducting charge carriers, respectively. The resulting set of EOM for the light-matter field arising from this Lagrangian couple Maxwell's equations for the four-potential and Schrödinger's equation for the order parameter,

$$\partial_\mu F^{\mu\nu} = \mu_0 \left(\mathcal{J}^\nu + j^\nu\right) \tag{2a}$$

$$i\hbar\dot{\psi} = \left(\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\mathbf{A}\right)^2 + qcA_0 \right)\psi, \tag{2b}$$

where $\mathcal{J}^\nu \equiv qn\,(c, \mathbf{v})^\nu$ is the four-current generated by superconducting charges with number density n and fluid velocity $\mathbf{v} \equiv (\hbar\boldsymbol{\nabla}\theta - q\mathbf{A})/m$. As derived in the Supplemental Material (SM) [20], the system's Hamiltonian can be expressed in an electrohydrodynamic form as

$$\mathcal{H} = \frac{\epsilon_0}{2}|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2 + n\left(\frac{1}{2}mv^2\right) + \frac{\hbar^2}{8m}n|\boldsymbol{\nabla}\ln n|^2, \tag{3}$$

with $\mathbf{E} = -c\boldsymbol{\nabla}A_0 - \dot{\mathbf{A}}$ the electric field, $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ the magnetic field, n the superconducting number density, and $v \equiv |\mathbf{v}|$ the fluid speed. Eq. (3) represents a decomposition of the total energy density into electric, magnetic, kinetic, and elastic components, respectively [21].

We now limit our focus to electrostatic systems, which are recovered by enforcing that all currents vanish $\boldsymbol{\mathcal{J}} = \mathbf{j} = \mathbf{0}$. We first introduce the bulk superconducting number density n_{s} and two important length scales: the London penetration depth $\lambda_{\mathrm{L}} = \sqrt{m/\left(\mu_0 q^2 n_{\mathrm{s}}\right)}$ and the Compton wavelength $\lambda_{\mathrm{C}} = h/(mc)$. In terms of the normalized number densities $\bar{n} \equiv n/n_{\mathrm{s}} = \mathcal{J}_0/(cqn_{\mathrm{s}})$ and $\bar{n}_{\mathrm{src}} \equiv -j_0/(cqn_{\mathrm{s}})$, Eqs. (2) reduce to the electrohydrostatic condition,

$$\bar{n} + 2\xi^4\nabla^2\frac{\nabla^2\sqrt{\bar{n}}}{\sqrt{\bar{n}}} = \bar{n}_{\mathrm{src}}, \tag{4}$$

revealing the healing length ξ given by

$$\xi \equiv \sqrt{\frac{\lambda_{\mathrm{L}}\lambda_{\mathrm{C}}}{4\pi}}. \tag{5}$$

As shown in the SM [20], Eq. (4) is a self-consistent statement of Gauss's law that expresses the balance between electric and elastic forces in the electrostatic distribution of the fluid: $q\mathbf{E} = \boldsymbol{\nabla} Q$ with $Q = -\hbar^2 \left(\nabla^2\sqrt{n}/\sqrt{n}\right)/(2m)$ the well-known quantum potential [22]. Because of the nonlinear term, proving the existence or uniqueness of solutions $\bar{n}$ is nontrivial and remains an open problem. Moreover, for general source distributions $\bar{n}_{\rm src}$, solutions are most attainable by numerical methods, which can exhibit instabilities stemming from the potential divergence of the nonlinear term as the density approaches zero. We may nonetheless make some qualitative observations regarding solutions to Eq. (4). First, we anticipate the asymptotic behavior $\bar{n} \to \bar{n}_{\rm src} = 1$ in the bulk. Second, spatial derivatives in the nonlinear term ensure C^4 continuity of $\bar{n}$ over all spatial coordinates. To avoid introducing additional length scales, we focus here on piecewise-constant sources $\bar{n}_{\rm src}$ that take values zero outside and one inside the superconducting material.

To obtain the electrohydrostatic pressure between two planar superconducting bodies, we first solve the electrohydrostatic condition sourced by two finitely separated ionic backgrounds. For each separation length $L \in [0, 20\xi]$, we then integrate the resulting electrohydrostatic energy density,

$$\mathcal{H} = \underbrace{\frac{\epsilon_0}{2}\left|\frac{\hbar^2}{2mq}\boldsymbol{\nabla}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right|^2}_{u_{\rm electric}} + \underbrace{\frac{\hbar^2}{8m}n|\boldsymbol{\nabla}\ln n|^2}_{u_{\rm elastic}}, \tag{6}$$

over all space V and compute the pressure $P = -\nabla_L \int_V \mathcal{H} dx$. Details of the calculation are provided in Fig. 1, with the main conclusion being the existence of a negative (attractive) pressure between plates that vanishes in the limit of zero or infinite separation and reaches a peak for $L \approx \xi$. Since C^4 continuity of the number density is guaranteed by the nonlocal quantum potential [23], all contributions to the electrohydrostatic energy density are finite. Consequently, unlike other quantum forces such as the Casimir pressure [24], the electrohydrostatic pressure does not exhibit a divergence for infinitesimal separations. Though the total pressure is strictly negative, the electric pressure exhibits a zero-crossing which can be understood perturbatively as a screening effect. As derived in the SM [20], for source distributions representing small perturbations from a uniform background, $\bar{n}_{\rm src} = 1 + \delta\bar{n}_{\rm src}$ with $|\delta\bar{n}_{\rm src}| \ll 1$, the electrohydrostatic condition reduces to a self-sourced version of the inhomogeneous biharmonic equation arising in linear elasticity theory [25],

$$\delta\bar{n} + \xi^4\nabla^4\delta\bar{n} \approx \delta\bar{n}_{\rm src}, \tag{7}$$

with $\delta\bar{n} \equiv \bar{n} - 1$ the first order perturbation in the number density and $\nabla^4 \equiv \nabla^2\nabla^2$ the biharmonic operator. In contrast to the Yukawa potential arising from Thomas-Fermi screening [26], the Green's function for Eq. (7),

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi\left(\xi\sqrt{2}\right)^3}\,\mathrm{sinc}\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\xi\sqrt{2}}\right)e^{-\frac{|\mathbf{x}-\mathbf{x}'|}{\xi\sqrt{2}}}, \tag{8}$$

exhibits both decaying and oscillatory behavior on the length scale of the healing length. The oscillatory component of the bulk response to a point source necessarily gives rise to interference effects during the screening of more general defect distributions. We can thus attribute increases (decreases) in electric energy density to constructive (destructive) interference of screening charges.

In the electrostatic limit, the healing length represents the scale on which the superconducting number density varies in response to changes in the background. While this interpretation might suggest analogies with the well-known GL coherence length, as seen from Eq. (5), the healing length and the London penetration depth are not independent parameters. As shown in the SM [20], the few known sources tabulating GL parameters from independent experiments indicate that our healing length and the GL coherence length are in poor agreement for most type-I superconductors but only differ by about one order of magnitude for many type-II superconductors [27]. This trend lends further support to the notion that the hydrodynamic model is likely most valid at temperatures far below the critical temperature, making type-II superconductors ideal candidates for experimental validation of the theory. Furthermore, since the healing length sets the scale underlying pressure variations, materials with large London penetration depths are desirable. For a conventional ($m = 2m_e$, $q = 2e$) superconductor with $\lambda_{\rm L} \approx 100$ nm, the pressure achieves a maximum value of ≈ 40 N/mm^2 for separations on the order of 1 Å.

The healing length can also be understood as the matter-like counterpart to the London penetration depth. As shown by way of perturbation theory in the SM [20], a uniform medium's first order response to a low-power drive supports the propagation of both longitudinal $\mathbf{k}_\parallel$ and transverse $\mathbf{k}_\perp$ plane waves in the fluid velocity field $\mathbf{v} \sim \mathbf{k}_{\parallel,\perp}e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\parallel,\perp}t)}$, with frequencies $\omega_{\parallel,\perp}(\mathbf{k})$ characterized by two different dispersions,

$$\omega_{||} = \omega_{\rm p}\sqrt{1 + (k\xi)^4} \tag{9a}$$

$$\omega_\perp = \omega_{\rm p}\sqrt{1 + (k\lambda_{\rm L})^2}, \tag{9b}$$

where $\omega_{\rm p} \equiv c/\lambda_{\rm L}$ is the plasma frequency and $k \equiv |\mathbf{k}|$ the wavenumber. The high-frequency limits $\omega_{\parallel,\perp} \gg \omega_{\rm p}$ of these relations manifest the longitudinal plane waves as matter-like polaritons $\omega_\parallel \approx \hbar k^2/(2m)$ and the transverse plane waves as light-like polaritons $\omega_\perp \approx ck$. With this insight, we can thus identify the quasistatic $\omega_{\parallel,\perp} \ll \omega_{\rm p}$ decay length of matter-like excitations with the healing length $\xi \approx 1/\,\mathrm{Im}[k(\omega_\parallel)]$ and light-like excitations with the London penetration depth $\lambda_{\rm L} \approx 1/\,\mathrm{Im}[k(\omega_\perp)]$.

To summarize the results of this study, we have presented a theory of superconductivity akin to the GL theory that is capable of describing the dynamics of superconducting quantum devices well below the critical temperature, and we have used the theory to predict a negative electrohydrostatic pressure between superconducting bodies. Moreover, we have identified an emergent healing length at which this pressure becomes relevant and shown that it is similar to the GL coherence length but represents the matter-like counterpart to the London penetration depth. This work naturally motivates an experimental demonstration of the pressure to determine the theory's validity, but the viability of such an observation requires understanding the magnitude of other forces present at this scale (e.g., Casimir and van der Waals interactions [24]), which is left to future work. Our formulation may also be applied to the analysis of magnetostatic systems, such as vortices, and dynamical systems, such as excited Josephson junctions. Finally, the theory may be further developed via second quantization and expanded to incorporate quasiparticle dynamics.

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Supplemental Material for
"Negative electrohydrostatic pressure between superconducting bodies"

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ELECTROHYDRODYNAMIC REPRESENTATION OF THE HAMILTONIAN

The following derivation assumes electrostatic sources $\mathbf{j} = \mathbf{0}$:

$$\begin{aligned}
\int \mathcal{H} d^3\mathbf{x} &= \int \left(\frac{\partial \mathcal{L}}{\partial \dot{A}_\rho} \dot{A}_\rho + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} \dot{\psi}^* - \mathcal{L} \right) d^3\mathbf{x} \\
&= \int \left(-\frac{1}{c\mu_0} F^{0\rho} \dot{A}_\rho + i\hbar \psi^* \dot{\psi} - \mathcal{L} \right) d^3\mathbf{x} \\
&= \int \left(-\frac{1}{c\mu_0} F^{0\rho} \dot{A}_\rho + \left(j_0 + cq|\psi|^2 \right) A_0 + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2m} \psi^* \left(\frac{\hbar}{i} \boldsymbol{\nabla} - q\mathbf{A} \right)^2 \psi \right) d^3\mathbf{x} \\
&= \int \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2 + \frac{1}{2m} \left| \left(\frac{\hbar}{i} \boldsymbol{\nabla} - q\mathbf{A} \right) \psi \right|^2 \right) d^3\mathbf{x} \\
&= \int \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2 + \frac{1}{2m} \left| \sqrt{n} \left(\hbar \boldsymbol{\nabla} \theta - q\mathbf{A} \right) + \frac{\hbar}{i} \boldsymbol{\nabla} \sqrt{n} \right|^2 \right) d^3\mathbf{x} \\
&= \int \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2 + n \left(\frac{1}{2} m v^2 \right) + \frac{\hbar^2}{8m} n |\boldsymbol{\nabla} \ln n|^2 \right) d^3\mathbf{x}.
\end{aligned} \tag{S1}$$

To arrive at the fourth line in Eq. (S1), we have employed Gauss's law and integration by parts. For more general source distributions, an analog of Poynting's theorem can be derived directly from the equations of motion,

$$\dot{\mathcal{H}} + \boldsymbol{\nabla} \cdot \mathbf{S} + \mathbf{j} \cdot \mathbf{E} = 0, \tag{S2}$$

where the directional energy flux in electrohydrodynamic form is given by

$$\mathbf{S} \equiv \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} + n\mathbf{v} \left(\frac{1}{2} m v^2 - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right) - \frac{\hbar^2}{4m} \dot{n} \boldsymbol{\nabla} \ln n. \tag{S3}$$

ELECTROHYDROSTATIC CONDITION

The equations of motion are invariant under the gauge transformation $(A^\mu, \theta) \to \left(A^\mu + \partial^\mu f, \theta - \frac{q}{\hbar} f \right)$ for any single-valued smooth function f, which motivates us to define the gauge-invariant four-potential $\mathcal{A}^\mu \equiv A^\mu + \frac{\hbar}{q} \partial^\mu \theta$. In terms of these variables, Maxwell's forms are preserved. Namely, the electric and magnetic fields are given by $\mathbf{E} = -c\boldsymbol{\nabla} \mathcal{A}_0 - \dot{\boldsymbol{\mathcal{A}}}$ and $\mathbf{B} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}$, respectively, and the electromagnetic tensor is given by $F^{\mu\nu} = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu$, which for notational consistency we now refer to as $\mathcal{F}^{\mu\nu} \equiv F^{\mu\nu}$. In terms of the gauge-invariant four-potential, Maxwell's equations thus undergo a trivial relabeling:

$$\partial_\mu F^{\mu\nu} = \mu_0 \left(\mathcal{J}^\nu + j^\nu \right) \implies \partial_\mu \mathcal{F}^{\mu\nu} = \mu_0 \left(\mathcal{J}^\nu + j^\nu \right). \tag{S4}$$

Since the fluid velocity $\mathbf{v} = -(q/m)\boldsymbol{\mathcal{A}}$, the superconducting four current may be written purely in terms of the number density and the gauge-invariant four-potential as $\mathcal{J}^\mu = qn \left(c, -(q/m)\boldsymbol{\mathcal{A}} \right)^\mu$. We proceed by expressing the imaginary and real parts of Schrödinger's equation in polar form in terms of the gauge-invariant four-potential, which correspond to the superconducting charge continuity equation and the quantum Hamilton-Jacobi equation, respectively:

$$\partial_\mu \mathcal{J}^\mu = 0 \implies \dot{n} = \frac{q}{m}\boldsymbol{\nabla}\cdot(n\boldsymbol{\mathcal{A}}) \tag{S5a}$$

$$-\hbar\dot{\theta} = \frac{1}{2}mv^2 - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} + qcA_0 \implies -qc\mathcal{A}_0 = \frac{q^2}{2m}|\boldsymbol{\mathcal{A}}|^2 - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}. \tag{S5b}$$

In an electrostatic system ($\mathbf{j} = \boldsymbol{\mathcal{A}} = \mathbf{0}$), the magnetic field vanishes and the electric field may be derived from the quantum Hamilton-Jacobi equation: $\mathbf{E} = -\hbar^2/(2mq)\boldsymbol{\nabla}\left(\nabla^2\sqrt{n}/\sqrt{n}\right)$. Moreover, the only nontrivial component of Maxwell's equations is Gauss's law, which reads $c\nabla^2\mathcal{A}_0 = -\left(qn + cj_0\right)/\epsilon_0$, and the only nontrivial component of Schrödinger's equation is the quantum Hamilton-Jacobi equation, which reads $c\mathcal{A}_0 = \hbar^2/(2mq)\nabla^2\sqrt{n}/\sqrt{n}$. Combining the two equations to eliminate $\mathcal{A}_0$, we arrive at the electrohydrostatic condition,

$$\frac{\hbar^2}{2mq}\nabla^2\frac{\nabla^2\sqrt{n}}{\sqrt{n}} = -\frac{1}{\epsilon_0}\left(qn + cj_0\right), \tag{S6}$$

which can be expressed in terms of the normalized number densities as given in the main text.

NUMERICAL DETAILS

The non-dimensionalized 1D electrohydrostatic condition and energy density employed in this study are given by

$$\bar{n} + 2\frac{d^2}{d\tilde{x}^2}\left(\frac{1}{\sqrt{\bar{n}}}\frac{d^2\sqrt{\bar{n}}}{d\tilde{x}^2}\right) = \Theta\left(\left|\tilde{x} - \tilde{L}/2\right|\right) \tag{S7a}$$

$$\int \mathcal{H}dx = u_0\xi\int\left(\underbrace{\left|\frac{d}{d\tilde{x}}\left(\frac{1}{\sqrt{\bar{n}}}\frac{d^2\sqrt{\bar{n}}}{d\tilde{x}^2}\right)\right|^2}_{u_{\text{electric}}\ [u_0]} + \underbrace{\frac{\bar{n}}{4}\left|\frac{d\ln\bar{n}}{d\tilde{x}}\right|^2}_{u_{\text{elastic}}\ [u_0]}\right)d\tilde{x}, \tag{S7b}$$

where $\tilde{x} \equiv x/\xi$, $\tilde{L} \equiv L/\xi$, and Θ is the Heaviside theta function. Calculation of the electrohydrostatic pressure between superconducting bodies was performed numerically using a central finite difference scheme with second-order accuracy for all spatial derivatives. The electrohydrostatic condition was solved on a numerical grid consisting of 801 equally spaced points $\tilde{x} \in [-20, 20]$ with boundary conditions $\bar{n}(\tilde{x})|_{|\tilde{x}|\geq 20} = 1$ for 201 equally spaced separation lengths $\tilde{L} \in [0, 20]$. Analytic manipulations of the electrohydrostatic condition were performed with SymPy, and the ensuing nonlinear vector equation was solved numerically using `scipy.optimize.fsolve`, which converged with the default tolerance of `1.49012e-08` for all results in this study.

SELF-SOURCED INHOMOGENEOUS BIHARMONIC EQUATION AND GREEN'S FUNCTION

As given in the main text, the electrohydrostatic condition reads

$$\bar{n} + 2\xi^4\nabla^2\frac{\nabla^2\sqrt{\bar{n}}}{\sqrt{\bar{n}}} = \bar{n}_{\text{src}}, \tag{S8}$$

which for a uniform background $\bar{n}_{\text{src}} = 1$ yields the trivial solution $\bar{n} = 1$. We derive the self-sourced version of the inhomogeneous biharmonic equation arising in linear elasticity theory by considering source distributions representing small perturbations from a uniform background as follows,

$$\bar{n}_{\text{src}}(\mathbf{x}) = 1 + \lambda\bar{n}^{(1)}_{\text{src}}(\mathbf{x}) \tag{S9a}$$

$$\bar{n}(\mathbf{x}) = 1 + \sum_{k=1}^{\infty}\lambda^k\bar{n}^{(k)}(\mathbf{x}), \tag{S9b}$$

so that as the perturbative parameter $\lambda \to 0$, we recover the unperturbed uniform medium. To first order in λ, the electrohydrostatic condition is given by

$$\begin{aligned} \lambda\bar{n}^{(1)} + \frac{\partial}{\partial\lambda}\left(2\xi^4\nabla^2\frac{\nabla^2\sqrt{\bar{n}}}{\sqrt{\bar{n}}}\right)\Bigg|_{\lambda=0}\lambda &= \lambda\bar{n}^{(1)}_{\text{src}} \\ \bar{n}^{(1)} + \xi^4\nabla^4\bar{n}^{(1)} &= \bar{n}^{(1)}_{\text{src}}, \end{aligned} \tag{S10}$$

as given in the main text with $\delta\bar{n}_{\rm src} \equiv \lambda\bar{n}_{\rm src}^{(1)}$ and $\delta\bar{n} \equiv \lambda\bar{n}^{(1)}$. To arrive at the second line, we have employed the quantum potential derivative identity derived in Eq. (S19). We now derive the corresponding Green's function by considering a source distribution of the form $\bar{n}_{\rm src}^{(1)}(\mathbf{x}) = \delta^{(3)}(\mathbf{x})$. Expanding $\bar{n}^{(1)}$ and $\bar{n}_{\rm src}^{(1)}$ in the Fourier basis,

$$\bar{n}^{(1)}(\mathbf{x}) \equiv \int \tilde{n}^{(1)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d^3\mathbf{k} \tag{S11a}$$

$$\bar{n}_{\rm src}^{(1)}(\mathbf{x}) \equiv \int \tilde{n}_{\rm src}^{(1)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d^3\mathbf{k}, \tag{S11b}$$

and taking the inverse Fourier transform of Eq. (S10) yields

$$\left(1 + (k\xi)^4\right)\tilde{n}^{(1)}(\mathbf{k}) = \tilde{n}_{\rm src}^{(1)}(\mathbf{k}) = \frac{1}{(2\pi)^3} \tag{S12}$$

with $k \equiv |\mathbf{k}|$. We proceed by solving for $\bar{n}^{(1)}$ in spherical coordinates with $\rho \equiv |\mathbf{x}|$:

$$\begin{aligned}
\bar{n}^{(1)}(\mathbf{x}) &= \frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{1+(k\xi)^4}d^3\mathbf{k} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi}\left(\frac{e^{ik\rho\cos\theta}}{1+(k\xi)^4}k^2\sin\theta\right)d\phi d\theta dk \\
&= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\pi\left(\frac{e^{ik\rho\cos\theta}}{1+(k\xi)^4}k^2\sin\theta\right)d\theta dk \\
&= \frac{1}{2\pi^2\rho}\int_0^\infty\left(\frac{k\sin(k\rho)}{1+(k\xi)^4}\right)dk \\
&= \frac{1}{4\pi\xi^2\rho}\sin\left(\frac{\rho}{\xi\sqrt{2}}\right)e^{-\frac{\rho}{\xi\sqrt{2}}}.
\end{aligned} \tag{S13}$$

For source distributions of the form $\bar{n}_{\rm src}^{(1)}(\mathbf{x}) = \delta^{(3)}(\mathbf{x}-\mathbf{x}')$, the first order response $\bar{n}^{(1)}(\mathbf{x})$ can be attained by a simple coordinate shift, which yields the Green's function in the main text. We note that the total net charge is neutral, as ensured by the property $\int_V \bar{n}^{(1)}d^3\mathbf{x} = \int_V \bar{n}_{\rm src}^{(1)}d^3\mathbf{x}$, where integration is performed over all space V.

HEALING LENGTH ANALOGS IN THE LITERATURE

Few sources in the literature tabulate superconducting parameters gathered from independent experiments. We include one such compilation [1] below and calculate the healing length from the main text using the reported London penetration depths. We find better agreement between our healing length and the GL coherence length for type-II superconductors than for type-I superconductors. Since the former tend to have higher critical temperatures than the latter, this trend lends further support to the notion that the hydrodynamic model is likely most valid at temperatures far below the critical temperature.

Type-I Superconductors						
Material	T_c [K]	Δ_0 [meV]	$\mu_0 H_{c0}$ [mT]	λ_0 [nm]	ξ_0 [nm]	$\sqrt{\frac{\lambda_0\lambda_C}{4\pi}}$ [nm]
Al	1.18	0.18	10.5	50	1600	0.07
In	3.41	0.54	23.0	65	360	0.08
Sn	3.72	0.59	30.5	50	230	0.07
Pb	7.20	1.35	80.0	40	90	0.06
Nb	9.25	1.50	198.0	85	40	0.09

TABLE S1. The penetration depth λ_0 and coherence length ξ_0 are given at zero temperature. Our healing length (last column) is calculated according to its definition in the main text with $\lambda_{\rm L} = \lambda_0$ and $\lambda_{\rm C}$ the Compton wavelength of a Cooper pair. *Source*: Donnelly, R. J. 1981. Cryogenics. In *Physics Vade Mecum*, ed. H. L. Anderson. American Institute of Physics, New York.

Type-II Superconductors						
Material	T_c [K]	Δ_0 [meV]	$\mu_0 H_{c0,2}$ [T]	$\lambda_{\mathrm{GL}}(0)$ [nm]	$\xi_{\mathrm{GL}}(0)$ [nm]	$\sqrt{\frac{\lambda_{\mathrm{GL}}(0)\lambda_{\mathrm{C}}}{4\pi}}$ [nm]
Pb-In	7.0	1.2	0.2	150	30	0.12
Pb-Bi	8.3	1.7	0.5	200	20	0.14
Nb-Ti	9.5	1.5	13	300	4	0.17
Nb-N	16	2.4	15	200	5	0.14
$PbMo_6S_8$	15	2.4	60	200	2	0.14
V_3Ga	15	2.3	23	90	2–3	0.09
V_3Si	16	2.3	20	60	3	0.08
Nb_3Sn	18	3.4	23	65	3	0.08
Nb_3Ge	23	3.7	38	90	3	0.09

TABLE S2. The temperature-dependent GL penetration depth and coherence length are given by $\lambda_{\mathrm{GL}}(T) = \lambda_{\mathrm{GL}}(0)(1 - T/T_c)^{-1/2}$ and $\xi_{\mathrm{GL}}(T) = \xi_{\mathrm{GL}}(0)(1 - T/T_c)^{-1/2}$, respectively. Our healing length (last column) is calculated according to its definition in the main text with $\lambda_{\mathrm{L}} = \lambda_{\mathrm{GL}}(0)$ and λ_{C} the Compton wavelength of a Cooper pair. We note that these values are only representative, since parameters for alloys and compounds depend on fabrication techniques. *Source*: Donnelly, R. J. 1981. Cryogenics. In *Physics Vade Mecum*, ed. H. L. Anderson. American Institute of Physics, New York.

POLARITON DISPERSION RELATIONS

The quantum Hamilton-Jacobi equation and Maxwell's equations can be combined to express Ampere's law with Maxwell's correction in terms of the gauge-invariant vector potential $\boldsymbol{\mathcal{A}}$ and the superconducting number density n:

$$\nabla \times \nabla \times \boldsymbol{\mathcal{A}} + \frac{1}{c^2}\ddot{\boldsymbol{\mathcal{A}}} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\left(-\frac{q}{2m}\left|\boldsymbol{\mathcal{A}}\right|^2 + \frac{\hbar^2}{2mq}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right) = \mu_0\left(-\frac{q^2}{m}n\boldsymbol{\mathcal{A}} + \mathbf{j}\right). \tag{S14}$$

To prepare for a perturbative analysis, we employ the quantum potential derivative identity derived in Eq. (S19) and the superconducting charge continuity equation to express the final term on the left side of Eq. (S14) in terms of dyadics [2]:

$$\frac{\hbar^2}{2qmc^2}\frac{\partial}{\partial t}\nabla\frac{\nabla^2\sqrt{n}}{\sqrt{n}} = \frac{\hbar^2}{4qmc^2}\nabla\left(n^{-3}[n,\nabla]^2\dot{n}\right) = \left(\frac{\lambda_{\mathrm{C}}}{4\pi}\right)^2\left\{n^{-3},\nabla\right\}[n,\nabla]^2\{n,\nabla\}\boldsymbol{\mathcal{A}}. \tag{S15}$$

We extract the system's low-power dynamics through the following perturbative expansion,

$$\mathbf{j}(\mathbf{x},t) = \lambda\mathbf{j}^{(1)}(\mathbf{x},t) \tag{S16a}$$

$$\boldsymbol{\mathcal{A}}(\mathbf{x},t) = \sum_{k=1}^{\infty}\lambda^k\boldsymbol{\mathcal{A}}^{(k)}(\mathbf{x},t) \tag{S16b}$$

$$n(\mathbf{x},t) = n^{(0)}(\mathbf{x}) + \sum_{k=1}^{\infty}\lambda^k n^{(k)}(\mathbf{x},t), \tag{S16c}$$

so that as the perturbative parameter $\lambda \to 0$, we recover the unperturbed electrostatic system. To first order in λ, Ampere's law with Maxwell's correction is given by

$$\left(\nabla \times \nabla \times + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\mu_0 q^2}{m}n^{(0)} + \left(\frac{\lambda_{\mathrm{C}}}{4\pi}\right)^2\left\{\left(n^{(0)}\right)^{-3},\nabla\right\}\left[n^{(0)},\nabla\right]^2\left\{n^{(0)},\nabla\right\}\right)\boldsymbol{\mathcal{A}}^{(1)} = \mu_0\mathbf{j}^{(1)}, \tag{S17}$$

which for a uniform medium with $n^{(0)} = n_{\mathrm{s}}$ reads

$$\left(\nabla\nabla - \nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{1}{\lambda_L^2} + \left(\frac{\lambda_{\mathrm{C}}}{4\pi}\right)^2\nabla^2\nabla\nabla\right)\boldsymbol{\mathcal{A}}^{(1)} = \mu_0\mathbf{j}^{(1)}. \tag{S18}$$

By employing longitudinal and transverse ansatzes $\boldsymbol{\mathcal{A}}^{(1)}_{\parallel,\perp} \sim \mathbf{k}_{\parallel,\perp}e^{i\left(\mathbf{k}\cdot\mathbf{x}-\omega_{\parallel,\perp}t\right)}$ and noting that $\nabla\nabla\boldsymbol{\mathcal{A}}^{(1)}_{\parallel} = -k^2\boldsymbol{\mathcal{A}}^{(1)}_{\parallel}$ and $\nabla\nabla\boldsymbol{\mathcal{A}}^{(1)}_{\perp} = \mathbf{0}$, it is straightforward to show that in regions with $\mathbf{j} = \mathbf{0}$, the system supports the propagation of longitudinal and transverse plane waves in the fluid velocity field with corresponding dispersion relations given in the main text.

QUANTUM POTENTIAL DERIVATIVE IDENTITY

The derivative of the quantum potential with respect to an arbitrary variable α can be written as follows,

$$\begin{aligned}\frac{\partial}{\partial\alpha}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} &= \frac{1}{2}\left(\frac{\nabla^2\frac{n'}{\sqrt{n}}}{\sqrt{n}} - \frac{\nabla^2\sqrt{n}}{n^{3/2}}n'\right) \\ &= \frac{1}{2\sqrt{n}}\left(\frac{1}{\sqrt{n}}\nabla^2 + 2\boldsymbol{\nabla}\frac{1}{\sqrt{n}}\cdot\boldsymbol{\nabla} + \nabla^2\frac{1}{\sqrt{n}} - \frac{\nabla^2\sqrt{n}}{n}\right)n' \\ &= \frac{1}{2}n^{-3}\left(n^2\nabla^2 - n\boldsymbol{\nabla}n\cdot\boldsymbol{\nabla} + |\boldsymbol{\nabla}n|^2 - n\nabla^2 n\right)n' \\ &= \frac{1}{2}n^{-3}[n,\boldsymbol{\nabla}]^2\frac{\partial n}{\partial\alpha},\end{aligned} \tag{S19}$$

where $n' \equiv \partial n/\partial\alpha$ [2].

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[1] K. Delin and T. Orlando, Superconductivity, in *The Engineering Handbook*, edited by R. C. Dorf (CRC Press, 1996) pp. 1274–1286.

[2] We employ the dyadic notation $[n,\boldsymbol{\nabla}]f \equiv (n\boldsymbol{\nabla})f - (\boldsymbol{\nabla}n)f$, $\{n,\boldsymbol{\nabla}\}f \equiv (n\boldsymbol{\nabla})f + (\boldsymbol{\nabla}n)f$, and $\boldsymbol{\nabla}\mathbf{f} \equiv \boldsymbol{\nabla}\cdot\mathbf{f}$.